JOURNAL OF FORMALIZED MATHEMATICS
Volume 11, Released 1999, Published 2003
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Noetherian Lattices

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Summary. In this article we define noetherian and co-noetherian lattices and show how some properties concerning upper and lower neighbours, irreducibility and density can be improved when restricted to these kinds of lattices. In addition we define atomic lattices.

MML Identifier: `LATTICE6`.

WWW: `http://mizar.org/JFM/Vol11/lattice6.html`

The articles [8], [4], [12], [13], [3], [14], [1], [5], [6], [11], [10], [2], [7], and [9] provide the notation and terminology for this paper.

Let us observe that there exists a lattice which is finite.

Let us observe that every lattice which is finite is also complete.

Let L be a lattice and let D be a subset of L. The functor $D^{\bullet}$ yields a subset of $\mathrm{Poset}(L)$ and is defined as follows:

(Def. 1) $D^{\bullet} = \{d^{\bullet}; d$ ranges over elements of L: $d \in D\}$.

Let L be a lattice and let D be a subset of $\mathrm{Poset}(L)$. The functor ${}^{\bullet}D$ yields a subset of L and is defined as follows:

(Def. 2) ${}^{\bullet}D = \{{}^{\bullet}d; d$ ranges over elements of $\mathrm{Poset}(L)$: $d \in D\}$.

Let L be a finite lattice. Note that $\mathrm{Poset}(L)$ is well founded.

Let L be a lattice. We say that L is noetherian if and only if:

(Def. 3) $\mathrm{Poset}(L)$ is well founded.

We say that L is co-noetherian if and only if:

(Def. 4) $\mathrm{Poset}(L)^{\smile}$ is well founded.

One can check that there exists a lattice which is noetherian, upper-bounded, lower-bounded, and complete.

One can verify that there exists a lattice which is co-noetherian, upper-bounded, lower-bounded, and complete.

One can prove the following proposition

(1) For every lattice L holds L is noetherian iff L° is co-noetherian.

Let us mention that every lattice which is finite is also noetherian and every lattice which is finite is also co-noetherian.

Let L be a lattice and let a, b be elements of L. We say that a is upper neighbour of b if and only if:

(Def. 5) $a \neq b$ and $b \sqsubseteq a$ and for every element c of L such that $b \sqsubseteq c$ and $c \sqsubseteq a$ holds $c = a$ or $c = b$.

We introduce b is lower neighbour of a as a synonym of a is upper neighbour of b.

One can prove the following propositions:

(2) Let L be a lattice, a be an element of L, and b, c be elements of L such that $b \neq c$. Then

(i) if b is upper neighbour of a and c is upper neighbour of a, then $a = c \sqcap b$, and

(ii) if b is lower neighbour of a and c is lower neighbour of a, then $a = c \sqcup b$.

(3) Let L be a noetherian lattice, a be an element of L, and d be an element of L. Suppose $a \sqsubseteq d$ and $a \neq d$. Then there exists an element c of L such that $c \sqsubseteq d$ and c is upper neighbour of a.

(4) Let L be a co-noetherian lattice, a be an element of L, and d be an element of L. Suppose $d \sqsubseteq a$ and $a \neq d$. Then there exists an element c of L such that $d \sqsubseteq c$ and c is lower neighbour of a.

(5) For every upper-bounded lattice L it is not true that there exists an element b of L such that b is upper neighbour of $\top_L$.

(6) Let L be a noetherian upper-bounded lattice and a be an element of L. Then $a = \top_L$ if and only if it is not true that there exists an element b of L such that b is upper neighbour of a.

(7) For every lower-bounded lattice L it is not true that there exists an element b of L such that b is lower neighbour of $\bot_L$.

(8) Let L be a co-noetherian lower-bounded lattice and a be an element of L. Then $a = \bot_L$ if and only if it is not true that there exists an element b of L such that b is lower neighbour of a.

Let L be a complete lattice and let a be an element of L. The functor $\overline{a}$ yielding an element of L is defined as follows:

(Def. 6) $\overline{a} = \bigsqcap_L \{d; d$ ranges over elements of L: $a \sqsubseteq d \wedge d \neq a\}$.

The functor ${}^*a$ yielding an element of L is defined by:

(Def. 7) ${}^*a = \bigsqcup_L \{d; d$ ranges over elements of L: $d \sqsubseteq a \wedge d \neq a\}$.

Let L be a complete lattice and let a be an element of L. We say that a is completely-meet-irreducible if and only if:

(Def. 8) $\overline{a} \neq a$.

We say that a is completely-join-irreducible if and only if:

(Def. 9) ${}^*a \neq a$.

One can prove the following propositions:

(9) For every complete lattice L and for every element a of L holds $a \sqsubseteq \overline{a}$ and ${}^*a \sqsubseteq a$.

(10) For every complete lattice L holds $\overline{\top_L} = \top_L$ and $(\top_L)^{\bullet}$ is meet-irreducible.

(11) For every complete lattice L holds ${}^*(\bot_L) = \bot_L$ and $(\bot_L)^{\bullet}$ is join-irreducible.

(12) Let L be a complete lattice and a be an element of L. Suppose a is completely-meet-irreducible. Then $\overline{a}$ is upper neighbour of a and for every element c of L such that c is upper neighbour of a holds $c = \overline{a}$.

(13) Let L be a complete lattice and a be an element of L. Suppose a is completely-join-irreducible. Then ${}^*a$ is lower neighbour of a and for every element c of L such that c is lower neighbour of a holds $c = {}^*a$.

(14) Let L be a noetherian complete lattice and a be an element of L. Then a is completely-meet-irreducible if and only if there exists an element b of L such that b is upper neighbour of a and for every element c of L such that c is upper neighbour of a holds $c = b$.

(15) Let L be a co-noetherian complete lattice and a be an element of L. Then a is completely-join-irreducible if and only if there exists an element b of L such that b is lower neighbour of a and for every element c of L such that c is lower neighbour of a holds $c = b$.

(16) Let L be a complete lattice and a be an element of L. If a is completely-meet-irreducible, then $a^{\bullet}$ is meet-irreducible.

(17) Let L be a complete noetherian lattice and a be an element of L. Suppose $a \neq \top_L$. Then a is completely-meet-irreducible if and only if $a^{\bullet}$ is meet-irreducible.

(18) Let L be a complete lattice and a be an element of L. If a is completely-join-irreducible, then $a^{\bullet}$ is join-irreducible.

(19) Let L be a complete co-noetherian lattice and a be an element of L. Suppose $a \neq \bot_L$. Then a is completely-join-irreducible if and only if $a^{\bullet}$ is join-irreducible.

(20) Let L be a finite lattice and a be an element of L such that $a \neq \bot_L$ and $a \neq \top_L$. Then
(i) a is completely-meet-irreducible iff $a^{\bullet}$ is meet-irreducible, and
(ii) a is completely-join-irreducible iff $a^{\bullet}$ is join-irreducible.

Let L be a lattice and let a be an element of L. We say that a is atomic if and only if:

(Def. 10) a is upper neighbour of $\bot_L$.

We say that a is co-atomic if and only if:

(Def. 11) a is lower neighbour of $\top_L$.

One can prove the following two propositions:

(21) Let L be a complete lattice and a be an element of L. If a is atomic, then a is completely-join-irreducible.

(22) Let L be a complete lattice and a be an element of L. If a is co-atomic, then a is completely-meet-irreducible.

Let L be a lattice. We say that L is atomic if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let a be an element of L. Then there exists a subset X of L such that for every element x of L such that $x \in X$ holds x is atomic and $a = \bigsqcup_L X$.

One can verify that there exists a lattice which is strict, non empty, and trivial.
Let us observe that there exists a lattice which is atomic and complete.
Let L be a complete lattice and let D be a subset of L. We say that D is supremum-dense if and only if:

(Def. 13) For every element a of L there exists a subset D' of D such that $a = \bigsqcup_L D'$.

We say that D is infimum-dense if and only if:

(Def. 14) For every element a of L there exists a subset D' of D such that $a = \bigsqcap_L D'$.

Next we state three propositions:

(23) Let L be a complete lattice and D be a subset of L. Then D is supremum-dense if and only if for every element a of L holds $a = \bigsqcup_L \{d; d$ ranges over elements of L: $d \in D \wedge d \sqsubseteq a\}$.

(24) Let L be a complete lattice and D be a subset of L. Then D is infimum-dense if and only if for every element a of L holds $a = \bigsqcap_L \{d; d$ ranges over elements of L: $d \in D \wedge a \sqsubseteq d\}$.

(25) Let L be a complete lattice and D be a subset of L. Then D is infimum-dense if and only if $D^{\bullet}$ is order-generating.

Let L be a complete lattice. The functor MIRRS L yields a subset of L and is defined by:

(Def. 15) MIRRS $L = \{a; a$ ranges over elements of L: a is completely-meet-irreducible$\}$.

The functor JIRRS L yields a subset of L and is defined by:

(Def. 16) JIRRS $L = \{a; a$ ranges over elements of L: a is completely-join-irreducible$\}$.

We now state two propositions:

(26) For every complete lattice L and for every subset D of L such that D is supremum-dense holds JIRRS $L \subseteq D$.

(27) For every complete lattice L and for every subset D of L such that D is infimum-dense holds MIRRS $L \subseteq D$.

Let L be a co-noetherian complete lattice. Observe that MIRRS L is infimum-dense.
Let L be a noetherian complete lattice. Observe that JIRRS L is supremum-dense.

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Received June 9, 1999

Published January 2, 2004