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# Homomorphisms of Lattices, Finite Join and Finite Meet

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The articles [11], [6], [14], [8], [15], [1], [4], [5], [13], [16], [12], [2], [9], [10], [7], and [3] provide the notation and terminology for this paper.

## 1. Preliminaries

In this paper $x$, $X$, $X_1$, $Y$, $Z$ are sets.

One can prove the following three propositions:

(1) If $\bigcup Y \subseteq Z$ and $X \in Y$, then $X \subseteq Z$.

(2) $\bigcup(X \Cap Y) = \bigcup X \cap \bigcup Y$.

(3) Let given $X$. Suppose that

(i) $X \neq \emptyset$, and

(ii) for every $Z$ such that $Z \neq \emptyset$ and $Z \subseteq X$ and $Z$ is $\subseteq$-linear there exists $Y$ such that $Y \in X$ and for every $X_1$ such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

Then there exists $Y$ such that $Y \in X$ and for every $Z$ such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

## 2. Lattice Theory

We adopt the following rules: $L$ is a lattice, $F$, $H$ are filters of $L$, and $p$, $q$, $r$ are elements of $L$.

We now state three propositions:

(4) $[L)$ is prime.

(5) $F \subseteq [F \cup H)$ and $H \subseteq [F \cup H)$.

(6) If $p \in [[q) \cup F)$, then there exists $r$ such that $r \in F$ and $q \sqcap r \sqsubseteq p$.

We follow the rules: $L_1$, $L_2$ denote lattices, $a_1$, $b_1$ denote elements of $L_1$, and $a_2$ denotes an element of $L_2$.

Let us consider $L_1$, $L_2$. A function from the carrier of $L_1$ into the carrier of $L_2$ is said to be a homomorphism from $L_1$ to $L_2$ if:

(Def. 1) $\mathrm{It}(a_1 \sqcup b_1) = \mathrm{it}(a_1) \sqcup \mathrm{it}(b_1)$ and $\mathrm{it}(a_1 \sqcap b_1) = \mathrm{it}(a_1) \sqcap \mathrm{it}(b_1)$.

In the sequel $f$ denotes a homomorphism from $L_1$ to $L_2$.
We now state the proposition

(7) If $a_1 \sqsubseteq b_1$, then $f(a_1) \sqsubseteq f(b_1)$.

Let us consider $L_1$, $L_2$ and let $f$ be a function from the carrier of $L_1$ into the carrier of $L_2$. We say that $f$ is monomorphism if and only if:

(Def. 2) $f$ is one-to-one.

We say that $f$ is epimorphism if and only if:

(Def. 3) $\mathrm{rng}\, f =$ the carrier of $L_2$.

The following propositions are true:

(8) If $f$ is monomorphism, then $a_1 \sqsubseteq b_1$ iff $f(a_1) \sqsubseteq f(b_1)$.

(9) Let $f$ be a function from the carrier of $L_1$ into the carrier of $L_2$. If $f$ is epimorphism, then for every $a_2$ there exists $a_1$ such that $a_2 = f(a_1)$.

Let us consider $L_1$, $L_2$, $f$. We say that $f$ is isomorphism if and only if:

(Def. 4) $f$ is monomorphism and epimorphism.

Let us consider $L_1$, $L_2$. Let us observe that $L_1$ and $L_2$ are isomorphic if and only if:

(Def. 5) There exists $f$ which is isomorphism.

Let us consider $L_1$, $L_2$, $f$. We say that $f$ preserves implication if and only if:

(Def. 6) $f(a_1 \Rightarrow b_1) = f(a_1) \Rightarrow f(b_1)$.

We say that $f$ preserves top if and only if:

(Def. 7) $f(\top_{(L_1)}) = \top_{(L_2)}$.

We say that $f$ preserves bottom if and only if:

(Def. 8) $f(\bot_{(L_1)}) = \bot_{(L_2)}$.

We say that $f$ preserves complement if and only if:

(Def. 9) $f({a_1}^{\mathrm{c}}) = f(a_1)^{\mathrm{c}}$.

Let us consider $L$. A subset of $L$ is called a closed subset of $L$ if:

(Def. 10) If $p \in$ it and $q \in$ it, then $p \sqcap q \in$ it and $p \sqcup q \in$ it.

One can prove the following proposition

(10) The carrier of $L$ is a closed subset of $L$.

Let us consider $L$. One can verify that there exists a closed subset of $L$ which is non empty.
One can prove the following proposition

(11) Every filter of $L$ is a closed subset of $L$.

In the sequel $B$ denotes a finite subset of the carrier of $L$.
Let us consider $L$, $B$. The functor $\bigsqcup^{\mathrm{f}}_B$ yields an element of $L$ and is defined as follows:

(Def. 12)[1] $\bigsqcup^{\mathrm{f}}_B = \bigsqcup^{\mathrm{f}}_B(\mathrm{id}_L)$.

The functor $\bigsqcap^{\mathrm{f}}_B$ yields an element of $L$ and is defined as follows:

[1] The definition (Def. 11) has been removed.

(Def. 13) $\bigsqcap^{\rm f}_B = \bigsqcap^{\rm f}_B(\mathrm{id}_L)$.

One can prove the following propositions:

(14)[2] $\bigsqcap^{\rm f}_B =$ (the meet operation of $L$)-$\sum_B \mathrm{id}_L$.

(15) $\bigsqcup^{\rm f}_B =$ (the join operation of $L$)-$\sum_B \mathrm{id}_L$.

(16) $\bigsqcup^{\rm f}_{\{p\}} = p$.

(17) $\bigsqcap^{\rm f}_{\{p\}} = p$.

## 3. Distributive Lattices

In the sequel $D_1$ denotes a distributive lattice and $f$ denotes a homomorphism from $D_1$ to $L_2$.

One can prove the following proposition

(18) If $f$ is epimorphism, then $L_2$ is distributive.

## 4. Lower-bounded Lattices

We use the following convention: $\ell_1$ denotes a lower-bounded lattice, $B$, $B_1$, $B_2$ denote finite subsets of the carrier of $\ell_1$, and $b$ denotes an element of $\ell_1$.

One can prove the following proposition

(19) Let $f$ be a homomorphism from $\ell_1$ to $L_2$. If $f$ is epimorphism, then $L_2$ is lower-bounded and $f$ preserves bottom.

In the sequel $f$ denotes a unary operation on the carrier of $\ell_1$.

One can prove the following propositions:

(20) $\bigsqcup^{\rm f}_{B\cup\{b\}} f = \bigsqcup^{\rm f}_B f \sqcup f(b)$.

(21) $\bigsqcup^{\rm f}_{B\cup\{b\}} = \bigsqcup^{\rm f}_B \sqcup b$.

(22) $\bigsqcup^{\rm f}_{(B_1)} \sqcup \bigsqcup^{\rm f}_{(B_2)} = \bigsqcup^{\rm f}_{B_1\cup B_2}$.

(23) $\bigsqcup^{\rm f}_{\emptyset_{\text{the carrier of } \ell_1}} = \bot_{(\ell_1)}$.

(24) For every closed subset $A$ of $\ell_1$ such that $\bot_{(\ell_1)} \in A$ and for every $B$ such that $B \subseteq A$ holds $\bigsqcup^{\rm f}_B \in A$.

## 5. Upper-bounded Lattices

We use the following convention: $\ell_2$ is an upper-bounded lattice, $B$, $B_1$, $B_2$ are finite subsets of the carrier of $\ell_2$, and $b$ is an element of $\ell_2$.

Next we state two propositions:

(25) For every homomorphism $f$ from $\ell_2$ to $L_2$ such that $f$ is epimorphism holds $L_2$ is upper-bounded and $f$ preserves top.

(26) $\bigsqcap^{\rm f}_{\emptyset_{\text{the carrier of } \ell_2}} = \top_{(\ell_2)}$.

In the sequel $f$, $g$ are unary operations on the carrier of $\ell_2$.

We now state several propositions:

(27) $\bigsqcap^{\rm f}_{B\cup\{b\}} f = \bigsqcap^{\rm f}_B f \sqcap f(b)$.

[2] The propositions (12) and (13) have been removed.

(28) $\bigsqcap^{\rm f}_{B \cup \{b\}} = \bigsqcap^{\rm f}_{B} \sqcap b$.

(29) $\bigsqcap^{\rm f}_{f^{\circ} B} g = \bigsqcap^{\rm f}_{B} (g \cdot f)$.

(30) $\bigsqcap^{\rm f}_{(B_1)} \sqcap \bigsqcap^{\rm f}_{(B_2)} = \bigsqcap^{\rm f}_{B_1 \cup B_2}$.

(31) For every closed subset $F$ of $\ell_2$ such that $\top_{(\ell_2)} \in F$ and for every $B$ such that $B \subseteq F$ holds $\bigsqcap^{\rm f}_{B} \in F$.

## 6. Distributive Upper-bounded Lattices

In the sequel $D_1$ is a distributive upper-bounded lattice, $B$ is a finite subset of the carrier of $D_1$, and $p$ is an element of $D_1$.

Next we state the proposition

(32) $\bigsqcap^{\rm f}_{B} \sqcup p = \bigsqcap^{\rm f}_{((\text{the join operation of } D_1)^{\circ}({\rm id}_{(D_1)}, p))^{\circ} B}$.

## 7. Implicative Lattices

For simplicity, we use the following convention: $C_1$ denotes a complemented lattice, $I_1$ denotes an implicative lattice, $f$ denotes a homomorphism from $I_1$ to $C_1$, and $i$, $j$, $k$ denote elements of $I_1$.

The following three propositions are true:

(33) $f(i) \sqcap f(i \Rightarrow j) \sqsubseteq f(j)$.

(34) If $f$ is monomorphism, then if $f(i) \sqcap f(k) \sqsubseteq f(j)$, then $f(k) \sqsubseteq f(i \Rightarrow j)$.

(35) If $f$ is isomorphism, then $C_1$ is implicative and $f$ preserves implication.

## 8. Boolean Lattices

For simplicity, we adopt the following rules: $B_3$ denotes a Boolean lattice, $f$ denotes a homomorphism from $B_3$ to $C_1$, $A$ denotes a non empty subset of $B_3$, $a$, $b$, $c$, $p$, $q$ denote elements of $B_3$, and $B$, $B_0$ denote finite subsets of the carrier of $B_3$.

We now state three propositions:

(36) $(\top_{(B_3)})^{\rm c} = \bot_{(B_3)}$.

(37) $(\bot_{(B_3)})^{\rm c} = \top_{(B_3)}$.

(38) If $f$ is epimorphism, then $C_1$ is Boolean and $f$ preserves complement.

Let us consider $B_3$. A non empty subset of $B_3$ is said to be a field of subsets of $B_3$ if:

(Def. 14) If $a \in$ it and $b \in$ it, then $a \sqcap b \in$ it and $a^{\rm c} \in$ it.

In the sequel $F$ is a field of subsets of $B_3$.

The following propositions are true:

(39) If $a \in F$ and $b \in F$, then $a \sqcup b \in F$.

(40) If $a \in F$ and $b \in F$, then $a \Rightarrow b \in F$.

(41) The carrier of $B_3$ is a field of subsets of $B_3$.

(42) $F$ is a closed subset of $B_3$.

Let us consider $B_3$, $A$. The field by $A$ yields a field of subsets of $B_3$ and is defined as follows:

(Def. 15) $A \subseteq$ the field by $A$ and for every $F$ such that $A \subseteq F$ holds the field by $A \subseteq F$.

Let us consider $B_3$, $A$. The functor $\mathrm{SetImp}(A)$ yields a subset of $B_3$ and is defined by:

(Def. 16) $\mathrm{SetImp}(A) = \{a \Rightarrow b : a \in A \wedge b \in A\}$.

Let us consider $B_3$, $A$. Note that $\mathrm{SetImp}(A)$ is non empty.
One can prove the following two propositions:

(43) $x \in \mathrm{SetImp}(A)$ iff there exist $p$, $q$ such that $x = p \Rightarrow q$ and $p \in A$ and $q \in A$.

(44) $c \in \mathrm{SetImp}(A)$ iff there exist $p$, $q$ such that $c = p^{\mathrm{c}} \sqcup q$ and $p \in A$ and $q \in A$.

Let us consider $B_3$. The functor $\mathrm{comp}\, B_3$ yields a function from the carrier of $B_3$ into the carrier of $B_3$ and is defined by:

(Def. 17) $(\mathrm{comp}\, B_3)(a) = a^{\mathrm{c}}$.

One can prove the following propositions:

(45) $\bigsqcup^{\mathrm{f}}_{B \cup \{b\}} \mathrm{comp}\, B_3 = \bigsqcup^{\mathrm{f}}_{B} \mathrm{comp}\, B_3 \sqcup b^{\mathrm{c}}$.

(46) $(\bigsqcup^{\mathrm{f}}_{B})^{\mathrm{c}} = \bigsqcap^{\mathrm{f}}_{B} \mathrm{comp}\, B_3$.

(47) $\bigsqcap^{\mathrm{f}}_{B \cup \{b\}} \mathrm{comp}\, B_3 = \bigsqcap^{\mathrm{f}}_{B} \mathrm{comp}\, B_3 \sqcap b^{\mathrm{c}}$.

(48) $(\bigsqcap^{\mathrm{f}}_{B})^{\mathrm{c}} = \bigsqcup^{\mathrm{f}}_{B} \mathrm{comp}\, B_3$.

(49) Let $A_1$ be a non empty closed subset of $B_3$. Suppose $\bot_{(B_3)} \in A_1$ and $\top_{(B_3)} \in A_1$. Let given $B$. If $B \subseteq \mathrm{SetImp}(A_1)$, then there exists $B_0$ such that $B_0 \subseteq \mathrm{SetImp}(A_1)$ and $\bigsqcup^{\mathrm{f}}_{B} \mathrm{comp}\, B_3 = \bigsqcap^{\mathrm{f}}_{(B_0)}$.

(50) For every non empty closed subset $A_1$ of $B_3$ such that $\bot_{(B_3)} \in A_1$ and $\top_{(B_3)} \in A_1$ holds $\{\bigsqcap^{\mathrm{f}}_{B} : B \subseteq \mathrm{SetImp}(A_1)\} =$ the field by $A_1$.

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