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On the Kuratowski Limit Operators[1]

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Summary. In the paper we give formal descriptions of the two Kuratowski limit oprators: Li S and Ls S, where S is an arbitrary sequence of subsets of a fixed topological space. In the two last sections we prove basic properties of these lower and upper topological limits, which may be found e.g. in [19]. In the sections 2–4, we present three operators which are associated in some sense with the above mentioned, that is lim inf F, lim sup F, and limes F, where F is a sequence of subsets of a fixed 1-sorted structure.

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The articles [29], [33], [2], [32], [9], [1], [22], [24], [35], [12], [34], [6], [4], [18], [8], [7], [16], [5], [13], [25], [30], [21], [10], [23], [14], [15], [20], [17], [27], [28], [26], [11], [3], and [31] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all sets X, x and for every subset A of X such that $x \notin A$ and $x \in X$ holds $x \in A^{\rm c}$.

(2) For every function F and for every set i such that $i \in \mathrm{dom} F$ holds $\bigcap F \subseteq F(i)$.

(3) Let T be a non empty 1-sorted structure and S_1, S_2 be sequences of subsets of the carrier of T. Then $S_1 = S_2$ if and only if for every natural number n holds $S_1(n) = S_2(n)$.

(4) For all sets A, B, C, D such that A meets B and C meets D holds $[:A, C:]$ meets $[:B, D:]$.

Let X be a 1-sorted structure. Observe that every sequence of subsets of the carrier of X is non empty.

Let T be a non empty 1-sorted structure. Note that there exists a sequence of subsets of the carrier of T which is non-empty.

Let T be a non empty 1-sorted structure. A sequence of subsets of T is a sequence of subsets of the carrier of T.

In this article we present several logical schemes. The scheme *LambdaSSeq* deals with a non empty 1-sorted structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

> There exists a sequence f of subsets of $\mathcal{A}$ such that for every natural number n holds $f(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExTopStrSeq* deals with a non empty topological space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

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There exists a sequence S of subsets of the carrier of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Next we state the proposition

(5) Let X be a non empty 1-sorted structure and F be a sequence of subsets of the carrier of X. Then $\operatorname{rng} F$ is a family of subsets of X.

Let X be a non empty 1-sorted structure and let F be a sequence of subsets of the carrier of X. Then $\bigcup F$ is a subset of X. Then $\bigcap F$ is a subset of X.

2. Lower and Upper Limit of Sequences of Subsets

Let X be a non empty set, let S be a function from $\mathbb{N}$ into X, and let k be a natural number. The functor $S \uparrow k$ yields a function from $\mathbb{N}$ into X and is defined as follows:

(Def. 2)[1] For every natural number n holds $(S \uparrow k)(n) = S(n+k)$.

Let X be a non empty 1-sorted structure and let F be a sequence of subsets of the carrier of X. The functor $\liminf F$ yields a subset of X and is defined as follows:

(Def. 3) There exists a sequence f of subsets of X such that $\liminf F = \bigcup f$ and for every natural number n holds $f(n) = \bigcap (F \uparrow n)$.

The functor $\limsup F$ yields a subset of X and is defined as follows:

(Def. 4) There exists a sequence f of subsets of X such that $\limsup F = \bigcap f$ and for every natural number n holds $f(n) = \bigcup (F \uparrow n)$.

We now state a number of propositions:

(6) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X, and x be a set. Then $x \in \bigcap F$ if and only if for every natural number z holds $x \in F(z)$.

(7) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X, and x be a set. Then $x \in \liminf F$ if and only if there exists a natural number n such that for every natural number k holds $x \in F(n+k)$.

(8) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X, and x be a set. Then $x \in \limsup F$ if and only if for every natural number n there exists a natural number k such that $x \in F(n+k)$.

(9) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\liminf F \subseteq \limsup F$.

(10) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\bigcap F \subseteq \liminf F$.

(11) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\limsup F \subseteq \bigcup F$.

(12) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\liminf F = (\limsup \operatorname{Complement} F)^{c}$.

(13) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cap B(n)$, then $\liminf C = \liminf A \cap \liminf B$.

(14) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cup B(n)$, then $\limsup C = \limsup A \cup \limsup B$.

[1] The definition (Def. 1) has been removed.

(15) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cup B(n)$, then $\liminf A \cup \liminf B \subseteq \liminf C$.

(16) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cap B(n)$, then $\limsup C \subseteq \limsup A \cap \limsup B$.

(17) Let X be a non empty 1-sorted structure, A be a sequence of subsets of the carrier of X, and B be a subset of X. If for every natural number n holds $A(n) = B$, then $\limsup A = B$.

(18) Let X be a non empty 1-sorted structure, A be a sequence of subsets of the carrier of X, and B be a subset of X. If for every natural number n holds $A(n) = B$, then $\liminf A = B$.

(19) Let X be a non empty 1-sorted structure, A, B be sequences of subsets of the carrier of X, and C be a subset of X. If for every natural number n holds $B(n) = C \dot{-} A(n)$, then $C \dot{-} \liminf A \subseteq \limsup B$.

(20) Let X be a non empty 1-sorted structure, A, B be sequences of subsets of the carrier of X, and C be a subset of X. If for every natural number n holds $B(n) = C \dot{-} A(n)$, then $C \dot{-} \limsup A \subseteq \limsup B$.

3. Ascending and Descending Families of Subsets

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T. We say that S is descending if and only if:

(Def. 5) For every natural number i holds $S(i+1) \subseteq S(i)$.

We say that S is ascending if and only if:

(Def. 6) For every natural number i holds $S(i) \subseteq S(i+1)$.

We now state several propositions:

(21) Let f be a function. Suppose that for every natural number i holds $f(i+1) \subseteq f(i)$. Let i, j be natural numbers. If $i \leq j$, then $f(j) \subseteq f(i)$.

(22) Let T be a non empty 1-sorted structure and C be a sequence of subsets of T. Suppose C is descending. Let i, m be natural numbers. If $i \geq m$, then $C(i) \subseteq C(m)$.

(23) Let T be a non empty 1-sorted structure and C be a sequence of subsets of T. Suppose C is ascending. Let i, m be natural numbers. If $i \geq m$, then $C(m) \subseteq C(i)$.

(24) Let T be a non empty 1-sorted structure, F be a sequence of subsets of T, and x be a set. Suppose F is descending and there exists a natural number k such that for every natural number n such that $n > k$ holds $x \in F(n)$. Then $x \in \bigcap F$.

(25) Let T be a non empty 1-sorted structure and F be a sequence of subsets of T. If F is descending, then $\liminf F = \bigcap F$.

(26) Let T be a non empty 1-sorted structure and F be a sequence of subsets of T. If F is ascending, then $\limsup F = \bigcup F$.

4. Constant and Convergent Sequences

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T. We say that S is convergent if and only if:

(Def. 7) $\limsup S = \liminf S$.

Next we state the proposition

(27) Let T be a non empty 1-sorted structure and S be a sequence of subsets of T. If S is constant, then the value of S is a subset of T.

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T. Let us observe that S is constant if and only if:

(Def. 8) There exists a subset A of T such that for every natural number n holds $S(n) = A$.

Let T be a non empty 1-sorted structure. Note that every sequence of subsets of T which is constant is also convergent, ascending, and descending.

Let T be a non empty 1-sorted structure. Note that there exists a sequence of subsets of T which is constant and non empty.

Let T be a non empty 1-sorted structure and let S be a convergent sequence of subsets of T. The functor limes S yields a subset of T and is defined by:

(Def. 9) limes $S = \limsup S$ and limes $S = \liminf S$.

One can prove the following proposition

(28) Let X be a non empty 1-sorted structure, F be a convergent sequence of subsets of X, and x be a set. Then $x \in$ limes F if and only if there exists a natural number n such that for every natural number k holds $x \in F(n+k)$.

5. Topological Lemmas

In the sequel n is a natural number.

Let f be a finite sequence of elements of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Note that $\widetilde{\mathcal{L}}(f)$ is closed.

One can prove the following propositions:

(29) Let r be a real number, M be a non empty Reflexive metric structure, and x be an element of M. If $0 < r$, then $x \in \mathrm{Ball}(x,r)$.

(30) For every point x of $\mathcal{E}^n$ and for every real number r holds $\mathrm{Ball}(x,r)$ is an open subset of $\mathcal{E}^n_{\mathrm{T}}$.

(31) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ and for all points p', q' of $\mathcal{E}^n$ such that $p = p'$ and $q = q'$ holds $\rho(p',q') = |p-q|$.

(32) Let p be a point of $\mathcal{E}^n$, x, p' be points of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $p = p'$ and $x \in \mathrm{Ball}(p,r)$, then $|x-p'| < r$.

(33) Let p be a point of $\mathcal{E}^n$, x, p' be points of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $p = p'$ and $|x-p'| < r$, then $x \in \mathrm{Ball}(p,r)$.

(34) Let n be a natural number, r be a point of $\mathcal{E}^n_{\mathrm{T}}$, and X be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $r \in \overline{X}$. Then there exists a sequence s_1 in $\mathcal{E}^n_{\mathrm{T}}$ such that $\mathrm{rng}\, s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 = r$.

Let M be a non empty metric space. Note that M_{top} is first-countable.

Let n be a natural number. Observe that $\mathcal{E}^n_{\mathrm{T}}$ is first-countable.

One can prove the following propositions:

(35) Let p be a point of $\mathcal{E}^n$, q be a point of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $p = q$ and $r > 0$, then $\mathrm{Ball}(p,r)$ is a neighbourhood of q.

(36) Let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$, p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and p' be a point of $\mathcal{E}^n$. Suppose $p = p'$. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p',r)$ meets A.

(37) Let x, y be points of $\mathcal{E}^n_{\mathrm{T}}$ and x' be a point of $\mathcal{E}^n$. If $x' = x$ and $x \neq y$, then there exists a real number r such that $y \notin \mathrm{Ball}(x',r)$.

(38) Let S be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then S is non Bounded if and only if for every real number r such that $r > 0$ there exist points x, y of $\mathcal{E}^n$ such that $x \in S$ and $y \in S$ and $\rho(x,y) > r$.

(39) For all real numbers a, b and for all points x, y of $\mathcal{E}^n$ such that $\mathrm{Ball}(x,a)$ meets $\mathrm{Ball}(y,b)$ holds $\rho(x,y) < a+b$.

(40) Let a, b, c be real numbers and x, y, z be points of $\mathcal{E}^n$. If $\mathrm{Ball}(x,a)$ meets $\mathrm{Ball}(z,c)$ and $\mathrm{Ball}(z,c)$ meets $\mathrm{Ball}(y,b)$, then $\rho(x,y) < a+b+2\cdot c$.

(41) Let X, Y be non empty topological spaces, x be a point of X, y be a point of Y, and V be a subset of $[\!: X, Y :\!]$. Then V is a neighbourhood of $[\!: \{x\}, \{y\} :\!]$ if and only if V is a neighbourhood of $\langle x, y \rangle$.

Now we present two schemes. The scheme *TSubsetEx* deals with a non empty topological structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a subset X of $\mathcal{A}$ such that for every point x of $\mathcal{A}$ holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

The scheme *TSubsetUniq* deals with a topological structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

Let A_1, A_2 be subsets of $\mathcal{A}$. Suppose for every point x of $\mathcal{A}$ holds $x \in A_1$ iff $\mathcal{P}[x]$ and for every point x of $\mathcal{A}$ holds $x \in A_2$ iff $\mathcal{P}[x]$. Then $A_1 = A_2$

for all values of the parameters.

Let T be a non empty topological structure, let S be a sequence of subsets of the carrier of T, and let i be a natural number. Then $S(i)$ is a subset of T.

We now state two propositions:

(42) Let T be a non empty 1-sorted structure, S be a sequence of subsets of the carrier of T, and R be a sequence of naturals. Then $S \cdot R$ is a sequence of subsets of T.

(43) $\mathrm{id}_{\mathbb{N}}$ is an increasing sequence of naturals.

Let us observe that $\mathrm{id}_{\mathbb{N}}$ is real-yielding.

6. SUBSEQUENCES

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of the carrier of T. A sequence of subsets of T is said to be a subsequence of S if:

(Def. 10) There exists an increasing sequence N_1 of naturals such that it $= S \cdot N_1$.

We now state several propositions:

(44) For every non empty 1-sorted structure T holds every sequence S of subsets of the carrier of T is a subsequence of S.

(45) Let T be a non empty 1-sorted structure, S be a sequence of subsets of T, and S_1 be a subsequence of S. Then $\mathrm{rng}\, S_1 \subseteq \mathrm{rng}\, S$.

(46) Let T be a non empty 1-sorted structure, S_1 be a sequence of subsets of the carrier of T, and S_2 be a subsequence of S_1. Then every subsequence of S_2 is a subsequence of S_1.

(47) Let T be a non empty 1-sorted structure, F, G be sequences of subsets of the carrier of T, and A be a subset of T. Suppose G is a subsequence of F and for every natural number i holds $F(i) = A$. Then $G = F$.

(48) Let T be a non empty 1-sorted structure, A be a constant sequence of subsets of T, and B be a subsequence of A. Then $A = B$.

(49) Let T be a non empty 1-sorted structure, S be a sequence of subsets of the carrier of T, R be a subsequence of S, and n be a natural number. Then there exists a natural number m such that $m \geq n$ and $R(n) = S(m)$.

Let T be a non empty 1-sorted structure and let X be a constant sequence of subsets of T. Observe that every subsequence of X is constant.

The scheme *SubSeqChoice* deals with a non empty topological space $\mathcal{A}$, a sequence $\mathcal{B}$ of subsets of the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

> There exists a subsequence S_1 of $\mathcal{B}$ such that for every natural number n holds $\mathcal{P}[S_1(n)]$

provided the parameters meet the following condition:

- For every natural number n there exists a natural number m such that $n \leq m$ and $\mathcal{P}[\mathcal{B}(m)]$.

7. The Lower Topological Limit

Let T be a non empty topological space and let S be a sequence of subsets of the carrier of T. The functor $\mathrm{Li}\, S$ yielding a subset of T is defined by the condition (Def. 11).

(Def. 11) Let p be a point of T. Then $p \in \mathrm{Li}\, S$ if and only if for every neighbourhood G of p there exists a natural number k such that for every natural number m such that $m > k$ holds $S(m)$ meets G.

One can prove the following propositions:

(50) Let S be a sequence of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and p' be a point of $\mathcal{E}^n$. Suppose $p = p'$. Then $p \in \mathrm{Li}\, S$ if and only if for every real number r such that $r > 0$ there exists a natural number k such that for every natural number m such that $m > k$ holds $S(m)$ meets $\mathrm{Ball}(p', r)$.

(51) For every non empty topological space T and for every sequence S of subsets of the carrier of T holds $\overline{\mathrm{Li}\, S} = \mathrm{Li}\, S$.

(52) For every non empty topological space T and for every sequence S of subsets of the carrier of T holds $\mathrm{Li}\, S$ is closed.

(53) Let T be a non empty topological space and R, S be sequences of subsets of the carrier of T. If R is a subsequence of S, then $\mathrm{Li}\, S \subseteq \mathrm{Li}\, R$.

(54) Let T be a non empty topological space and A, B be sequences of subsets of the carrier of T. If for every natural number i holds $A(i) \subseteq B(i)$, then $\mathrm{Li}\, A \subseteq \mathrm{Li}\, B$.

(55) Let T be a non empty topological space and A, B, C be sequences of subsets of the carrier of T. If for every natural number i holds $C(i) = A(i) \cup B(i)$, then $\mathrm{Li}\, A \cup \mathrm{Li}\, B \subseteq \mathrm{Li}\, C$.

(56) Let T be a non empty topological space and A, B, C be sequences of subsets of the carrier of T. If for every natural number i holds $C(i) = A(i) \cap B(i)$, then $\mathrm{Li}\, C \subseteq \mathrm{Li}\, A \cap \mathrm{Li}\, B$.

(57) Let T be a non empty topological space and F, G be sequences of subsets of the carrier of T. If for every natural number i holds $G(i) = \overline{F(i)}$, then $\mathrm{Li}\, G = \mathrm{Li}\, F$.

(58) Let S be a sequence of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Given a sequence s in $\mathcal{E}^n_{\mathrm{T}}$ such that s is convergent and for every natural number x holds $s(x) \in S(x)$ and $p = \lim s$. Then $p \in \mathrm{Li}\, S$.

(59) Let T be a non empty topological space, P be a subset of T, and s be a sequence of subsets of the carrier of T. If for every natural number i holds $s(i) \subseteq P$, then $\mathrm{Li}\, s \subseteq \overline{P}$.

(60) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T, and A be a subset of T. If for every natural number i holds $F(i) = A$, then $\mathrm{Li}\, F = \overline{A}$.

(61) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T, and A be a closed subset of T. If for every natural number i holds $F(i) = A$, then $\mathrm{Li}\, F = A$.

(62) Let S be a sequence of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is Bounded and for every natural number i holds $S(i) \subseteq P$. Then $\mathrm{Li}\, S$ is Bounded.

(63) Let S be a sequence of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is Bounded and for every natural number i holds $S(i) \subseteq P$ and for every natural number i holds $S(i)$ is compact. Then $\mathrm{Li}\, S$ is compact.

(64) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and C be a sequence of subsets of the carrier of $[:\mathcal{E}^n_{\mathrm{T}}, \mathcal{E}^n_{\mathrm{T}}:]$. If for every natural number i holds $C(i) = [:A(i), B(i):]$, then $[:\mathrm{Li}\, A, \mathrm{Li}\, B:] = \mathrm{Li}\, C$.

(65) For every sequence S of subsets of $\mathcal{E}^2_{\mathrm{T}}$ holds $\liminf S \subseteq \mathrm{Li}\, S$.

(66) For every simple closed curve C and for every natural number i holds $\mathrm{Fr}((\mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))^{\mathrm{c}}) = \widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))$.

8. The Upper Topological Limit

Let T be a non empty topological space and let S be a sequence of subsets of the carrier of T. The functor $\mathrm{Ls}\, S$ yields a subset of T and is defined as follows:

(Def. 12) For every set x holds $x \in \mathrm{Ls}\, S$ iff there exists a subsequence A of S such that $x \in \mathrm{Li}\, A$.

We now state a number of propositions:

(67) Let N be a natural number, F be a sequence of $\mathcal{E}^N_{\mathrm{T}}$, x be a point of $\mathcal{E}^N_{\mathrm{T}}$, and x' be a point of $\mathcal{E}^N$. Suppose $x = x'$. Then x is a cluster point of F if and only if for every real number r and for every natural number n such that $r > 0$ there exists a natural number m such that $n \leq m$ and $F(m) \in \mathrm{Ball}(x', r)$.

(68) For every non empty topological space T and for every sequence A of subsets of the carrier of T holds $\mathrm{Li}\, A \subseteq \mathrm{Ls}\, A$.

(69) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose for every natural number i holds $A(i) \subseteq B(i)$ and C is a subsequence of A. Then there exists a subsequence D of B such that for every natural number i holds $C(i) \subseteq D(i)$.

(70) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose for every natural number i holds $A(i) \subseteq B(i)$ and C is a subsequence of B. Then there exists a subsequence D of A such that for every natural number i holds $D(i) \subseteq C(i)$.

(71) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $A(i) \subseteq B(i)$, then $\mathrm{Ls}\, A \subseteq \mathrm{Ls}\, B$.

(72) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $C(i) = A(i) \cup B(i)$, then $\mathrm{Ls}\, A \cup \mathrm{Ls}\, B \subseteq \mathrm{Ls}\, C$.

(73) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $C(i) = A(i) \cap B(i)$, then $\mathrm{Ls}\, C \subseteq \mathrm{Ls}\, A \cap \mathrm{Ls}\, B$.

(74) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and C, C_1 be sequences of subsets of the carrier of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$. Suppose for every natural number i holds $C(i) = [:A(i), B(i):]$ and C_1 is a subsequence of C. Then there exist sequences A_1, B_1 of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that A_1 is a subsequence of A and B_1 is a subsequence of B and for every natural number i holds $C_1(i) = [:A_1(i), B_1(i):]$.

(75) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and C be a sequence of subsets of the carrier of $[:\mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}}:]$. If for every natural number i holds $C(i) = [:A(i), B(i):]$, then $\mathrm{Ls}\, C \subseteq [:\mathrm{Ls}\, A, \mathrm{Ls}\, B:]$.

(76) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T, and A be a subset of T. If for every natural number i holds $F(i) = A$, then $\operatorname{Li} F = \operatorname{Ls} F$.

(77) Let F be a sequence of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and A be a subset of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $F(i) = A$, then $\operatorname{Ls} F = \overline{A}$.

(78) Let F, G be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $G(i) = \overline{F(i)}$, then $\operatorname{Ls} G = \operatorname{Ls} F$.

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