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Inst. of Computer Science, Univ. of Białystok

On the Kuratowski Closure-Complement Problem

Lilla Krystyna Bagińska
University of Białystok

Adam Grabowski
University of Białystok

Summary. In this article we formalize the Kuratowski closure-complement result: there is at most 14 distinct sets that one can produce from a given subset A of a topological space T by applying closure and complement operators and that all 14 can be obtained from a suitable subset of $\mathbb{R}$, namely KuratExSet $= \{1\} \cup \mathbb{Q}(2,3) \cup (3,4) \cup (4,\infty)$.

The second part of the article deals with the maximal number of distinct sets which may be obtained from a given subset A of T by applying closure and interior operators. The subset KuratExSet of $\mathbb{R}$ is also enough to show that 7 can be achieved.

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The articles [13], [15], [14], [10], [16], [12], [1], [3], [11], [7], [6], [8], [2], [4], [9], and [5] provide the notation and terminology for this paper.

1. Fourteen Kuratowski Sets

In this paper T is a non empty topological space and A is a subset of T.

We now state the proposition

(1) $\overline{\overline{\overline{\overline{A}^{c}}^{c}}^{c}} = \overline{\overline{A}^{c}}$.

Let us consider T, A. The functor Kurat14Part(A) is defined as follows:

(Def. 1) $\text{Kurat14Part}(A) = \{A, \overline{A}, \overline{A}^{c}, \overline{\overline{A}^{c}}, \overline{\overline{A}^{c}}^{c}, \overline{\overline{\overline{A}^{c}}^{c}}, \overline{\overline{\overline{A}^{c}}^{c}}^{c}\}$.

Let us consider T, A. Observe that Kurat14Part(A) is finite.

Let us consider T, A. The functor Kurat14Set(A) yields a family of subsets of T and is defined by:

(Def. 2) $\text{Kurat14Set}(A) = \{A, \overline{A}, \overline{A}^{c}, \overline{\overline{A}^{c}}, \overline{\overline{A}^{c}}^{c}, \overline{\overline{\overline{A}^{c}}^{c}}, \overline{\overline{\overline{A}^{c}}^{c}}^{c}\} \cup \{A^{c}, \overline{A^{c}}, \overline{A^{c}}^{c}, \overline{\overline{A^{c}}^{c}}, \overline{\overline{A^{c}}^{c}}^{c}, \overline{\overline{\overline{A^{c}}^{c}}^{c}}, \overline{\overline{\overline{A^{c}}^{c}}^{c}}^{c}\}$.

Next we state three propositions:

(2) $\text{Kurat14Set}(A) = \text{Kurat14Part}(A) \cup \text{Kurat14Part}(A^{c})$.

(3) $A \in \text{Kurat14Set}(A)$ and $\overline{A} \in \text{Kurat14Set}(A)$ and $\overline{A}^{c} \in \text{Kurat14Set}(A)$ and $\overline{\overline{A}^{c}} \in \text{Kurat14Set}(A)$ and $\overline{\overline{A}^{c}}^{c} \in \text{Kurat14Set}(A)$ and $\overline{\overline{\overline{A}^{c}}^{c}} \in \text{Kurat14Set}(A)$ and $\overline{\overline{\overline{A}^{c}}^{c}}^{c} \in \text{Kurat14Set}(A)$.

(4) $A^{c} \in \text{Kurat14Set}(A)$ and $\overline{A^{c}} \in \text{Kurat14Set}(A)$ and $\overline{A^{c}}^{c} \in \text{Kurat14Set}(A)$ and $\overline{\overline{A^{c}}^{c}} \in \text{Kurat14Set}(A)$ and $\overline{\overline{A^{c}}^{c}}^{c} \in \text{Kurat14Set}(A)$ and $\overline{\overline{\overline{A^{c}}^{c}}^{c}} \in \text{Kurat14Set}(A)$ and $\overline{\overline{\overline{A^{c}}^{c}}^{c}}^{c} \in \text{Kurat14Set}(A)$.

Let us consider T, A. The functor Kurat14ClosedPart(A) yields a family of subsets of T and is defined by:

(Def. 3) $\quad \text{Kurat14ClosedPart}(A) = \{\overline{A}, \overline{\overline{A}^{c}}, \overline{\overline{\overline{A}^{c}}^{c}}, \overline{A^{c}}, \overline{\overline{A^{c}}^{c}}, \overline{\overline{\overline{A^{c}}^{c}}^{c}}\}.$

The functor Kurat14OpenPart(A) yields a family of subsets of T and is defined by:

(Def. 4) $\quad \text{Kurat14OpenPart}(A) = \{\overline{A}^{c}, \overline{\overline{A}^{c}}^{c}, \overline{\overline{\overline{A}^{c}}^{c}}^{c}, \overline{A^{c}}^{c}, \overline{\overline{A^{c}}^{c}}^{c}, \overline{\overline{\overline{A^{c}}^{c}}^{c}}^{c}\}.$

Next we state the proposition

(5) $\quad \text{Kurat14Set}(A) = \{A, A^{c}\} \cup \text{Kurat14ClosedPart}(A) \cup \text{Kurat14OpenPart}(A).$

Let us consider T, A. Observe that Kurat14Set(A) is finite.
Next we state two propositions:

(6) For every subset Q of T such that $Q \in \text{Kurat14Set}(A)$ holds $Q^{c} \in \text{Kurat14Set}(A)$ and $\overline{Q} \in \text{Kurat14Set}(A)$.

(7) $\quad \text{card}\,\text{Kurat14Set}(A) \leq 14.$

2. SEVEN KURATOWSKI SETS

Let us consider T, A. The functor Kurat7Set(A) yields a family of subsets of T and is defined by:

(Def. 5) $\quad \text{Kurat7Set}(A) = \{A, \text{Int}\,A, \overline{A}, \text{Int}\,\overline{A}, \overline{\text{Int}\,A}, \overline{\text{Int}\,\overline{A}}, \text{Int}\,\overline{\text{Int}\,A}\}.$

The following propositions are true:

(8) $A \in \text{Kurat7Set}(A)$ and $\text{Int}\,A \in \text{Kurat7Set}(A)$ and $\overline{A} \in \text{Kurat7Set}(A)$ and $\text{Int}\,\overline{A} \in \text{Kurat7Set}(A)$ and $\overline{\text{Int}\,A} \in \text{Kurat7Set}(A)$ and $\overline{\text{Int}\,\overline{A}} \in \text{Kurat7Set}(A)$ and $\text{Int}\,\overline{\text{Int}\,A} \in \text{Kurat7Set}(A)$.

(9) $\quad \text{Kurat7Set}(A) = \{A\} \cup \{\text{Int}\,A, \text{Int}\,\overline{A}, \text{Int}\,\overline{\text{Int}\,A}\} \cup \{\overline{A}, \overline{\text{Int}\,A}, \overline{\text{Int}\,\overline{A}}\}.$

Let us consider T, A. One can check that Kurat7Set(A) is finite.
We now state two propositions:

(10) For every subset Q of T such that $Q \in \text{Kurat7Set}(A)$ holds $\text{Int}\,Q \in \text{Kurat7Set}(A)$ and $\overline{Q} \in \text{Kurat7Set}(A)$.

(11) $\quad \text{card}\,\text{Kurat7Set}(A) \leq 7.$

3. THE SET GENERATING EXACTLY FOURTEEN KURATOWSKI SETS

The subset KuratExSet of $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 6) $\quad \text{KuratExSet} = \{1\} \cup]2,3[_{\mathbb{Q}} \cup]3,4[\cup]4,+\infty[.$

We now state a number of propositions:

(12) $\quad \overline{\text{KuratExSet}} = \{1\} \cup [2,+\infty[.$

(13) $\quad \overline{\text{KuratExSet}}^{c} =]-\infty,1[\cup]1,2[.$

(14) $\quad \overline{\overline{\text{KuratExSet}}^{c}} =]-\infty,2].$

(15) $\quad \overline{\overline{\text{KuratExSet}}^{c}}^{c} =]2,+\infty[.$

(16) $\quad \overline{\overline{\overline{\text{KuratExSet}}^{c}}^{c}} = [2,+\infty[.$

(17) $\quad \overline{\overline{\overline{\text{KuratExSet}}^{c}}^{c}}^{c} =]-\infty,2[.$

(18) $\mathrm{KuratExSet}^{\mathrm{c}} =]-\infty,1[\cup]1,2]\cup]2,3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(19) $\overline{\mathrm{KuratExSet}^{\mathrm{c}}} =]-\infty,3] \cup \{4\}$.

(20) $\overline{\mathrm{KuratExSet}^{\mathrm{c}}}^{\mathrm{c}} =]3,4[\cup]4,+\infty[$.

(21) $\overline{\overline{\mathrm{KuratExSet}^{\mathrm{c}}}^{\mathrm{c}}} = [3,+\infty[$.

(22) $\overline{\overline{\mathrm{KuratExSet}^{\mathrm{c}}}^{\mathrm{c}}}^{\mathrm{c}} =]-\infty,3[$.

(23) $\overline{\overline{\overline{\mathrm{KuratExSet}^{\mathrm{c}}}^{\mathrm{c}}}^{\mathrm{c}}} =]-\infty,3]$.

(24) $\overline{\overline{\overline{\mathrm{KuratExSet}^{\mathrm{c}}}^{\mathrm{c}}}^{\mathrm{c}}}^{\mathrm{c}} =]3,+\infty[$.

4. The Set Generating Exactly Seven Kuratowski Sets

One can prove the following propositions:

(25) $\mathrm{Int}\,\mathrm{KuratExSet} =]3,4[\cup]4,+\infty[$.

(26) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} = [3,+\infty[$.

(27) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} =]3,+\infty[$.

(28) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} =]2,+\infty[$.

(29) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} = [2,+\infty[$.

5. The Difference Between Chosen Kuratowski Sets

We now state a number of propositions:

(30) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\overline{\mathrm{KuratExSet}}$.

(31) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \overline{\mathrm{KuratExSet}}$.

(32) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(33) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(34) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(35) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(36) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(37) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(38) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(39) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(40) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(41) $\mathrm{Int}\,\mathrm{KuratExSet} \neq \overline{\mathrm{KuratExSet}}$.

(42) $\overline{\mathrm{KuratExSet}} \neq \mathrm{KuratExSet}$.

(43) $\mathrm{KuratExSet} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(44) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(45) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(46) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

6. Final Proofs For Seven Sets

The following propositions are true:

(47) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{KuratExSet}}$.

(48) $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$ are mutually different.

(49) $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

(50) For every set X such that $X \in \{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ holds X is an open non empty subset of $\mathbb{R}^{\mathbf{1}}$.

(51) For every set X such that $X \in \{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ holds X is a closed subset of $\mathbb{R}^{\mathbf{1}}$.

(52) For every set X such that $X \in \{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ holds $X \neq \mathbb{R}$.

(53) For every set X such that $X \in \{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ holds $X \neq \mathbb{R}$.

(54) $\{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ misses $\{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$.

(55) $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

Let us note that KuratExSet is non closed and non open.

One can prove the following propositions:

(56) $\{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ misses $\{\mathrm{KuratExSet}\}$.

(57) $\mathrm{KuratExSet}$, $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

(58) $\mathrm{card}\,\mathrm{Kurat7Set}(\mathrm{KuratExSet}) = 7$.

7. Final Proofs For Fourteen Sets

One can check that Kurat14ClosedPart(KuratExSet) has proper subsets and Kurat14OpenPart(KuratExSet) has proper subsets.

Let us observe that Kurat14Set(KuratExSet) has proper subsets.

One can verify that Kurat14Set(KuratExSet) has non empty elements.

One can prove the following proposition

(59) For every set A with non empty elements and for every set B such that $B \subseteq A$ holds B has non empty elements.

One can verify that Kurat14ClosedPart(KuratExSet) has non empty elements and Kurat14OpenPart(KuratExSet) has non empty elements.

Let us observe that there exists a family of subsets of $\mathbb{R}^{\mathbf{1}}$ which has proper subsets and non empty elements.

Next we state the proposition

(60) Let F, G be families of subsets of $\mathbb{R}^{\mathbf{1}}$ with proper subsets and non empty elements. If F is open and G is closed, then F misses G.

Let us observe that Kurat14ClosedPart(KuratExSet) is closed and Kurat14OpenPart(KuratExSet) is open.

Next we state the proposition

(61) Kurat14ClosedPart(KuratExSet) misses Kurat14OpenPart(KuratExSet).

Let us consider T, A. One can check that Kurat14ClosedPart(A) is finite and Kurat14OpenPart(A) is finite.

The following three propositions are true:

(62) card Kurat14ClosedPart(KuratExSet) $= 6$.

(63) card Kurat14OpenPart(KuratExSet) $= 6$.

(64) {KuratExSet, KuratExSet$^{\rm c}$} misses Kurat14ClosedPart(KuratExSet).

Let us observe that KuratExSet is non empty.

Next we state three propositions:

(65) KuratExSet $\neq$ KuratExSet$^{\rm c}$.

(66) {KuratExSet, KuratExSet$^{\rm c}$} misses Kurat14OpenPart(KuratExSet).

(67) card Kurat14Set(KuratExSet) $= 14$.

8. Properties of Kuratowski Sets

Let T be a topological structure and let A be a family of subsets of T. We say that A is closed for closure operator if and only if:

(Def. 7) For every subset P of T such that $P \in A$ holds $\overline{P} \in A$.

We say that A is closed for interior operator if and only if:

(Def. 8) For every subset P of T such that $P \in A$ holds $\mathrm{Int}\, P \in A$.

Let T be a 1-sorted structure and let A be a family of subsets of T. We say that A is closed for complement operator if and only if:

(Def. 9) For every subset P of T such that $P \in A$ holds $P^{\rm c} \in A$.

Let us consider T, A. One can check the following observations:

* Kurat14Set(A) is non empty,
* Kurat14Set(A) is closed for closure operator, and
* Kurat14Set(A) is closed for complement operator.

Let us consider T, A. One can verify the following observations:

* Kurat7Set(A) is non empty,
* Kurat7Set(A) is closed for interior operator, and
* Kurat7Set(A) is closed for closure operator.

Let us consider T. Note that there exists a family of subsets of T which is closed for interior operator, closed for closure operator, and non empty and there exists a family of subsets of T which is closed for complement operator, closed for closure operator, and non empty.

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