JOURNAL OF FORMALIZED MATHEMATICS
Volume 11, Released 1999, Published 2003
Inst. of Computer Science, Univ. of Białystok

Cages - the External Approximation of Jordan's Curve

Czesław Byliński
University of Białystok

Mariusz Żynel
University of Białystok

Summary. On the Euclidean plane Jordan's curve may be approximated with a polygonal path of sides parallel to coordinate axes, either externally, or internally. The paper deals with the external approximation, and the existence of a *Cage* – an external polygonal path – is proved.

MML Identifier: JORDAN9.

WWW: http://mizar.org/JFM/Vol11/jordan9.html

The articles [25], [30], [21], [3], [27], [2], [22], [31], [6], [7], [1], [4], [8], [29], [9], [18], [16], [17], [24], [32], [13], [23], [5], [14], [15], [19], [20], [12], [28], [26], [10], and [11] provide the notation and terminology for this paper.

1. Generalities

We adopt the following convention: i, j, k, n are natural numbers, D is a non empty set, and f, g are finite sequences of elements of D.

One can prove the following four propositions:

(3)[1] Let T be a non empty topological space and B, C_1, C_2, D be subsets of T. Suppose B is connected and C_1 is a component of D and C_2 is a component of D and B meets C_1 and B meets C_2 and $B \subseteq D$. Then $C_1 = C_2$.

(4) If for every n holds $f \restriction n = g \restriction n$, then $f = g$.

(5) If $n \in \mathrm{dom}\, f$, then there exists k such that $k \in \mathrm{dom}\,\mathrm{Rev}(f)$ and $n+k = \mathrm{len}\, f + 1$ and $f_n = (\mathrm{Rev}(f))_k$.

(6) If $n \in \mathrm{dom}\,\mathrm{Rev}(f)$, then there exists k such that $k \in \mathrm{dom}\, f$ and $n+k = \mathrm{len}\, f + 1$ and $(\mathrm{Rev}(f))_n = f_k$.

2. Go-Board Preliminaries

For simplicity, we adopt the following convention: G is a Go-board, f, g are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p is a point of $\mathcal{E}^2_{\mathrm{T}}$, r, s are real numbers, and x is a set.

We now state a number of propositions:

(7) Let D be a non empty set, G be a matrix over D, and f be a finite sequence of elements of D. Then f is a sequence which elements belong to G if and only if $\mathrm{Rev}(f)$ is a sequence which elements belong to G.

[1] The propositions (1) and (2) have been removed.

(8) Let G be a matrix over $\mathcal{E}^2_{\rm T}$ and f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a sequence which elements belong to G and $1 \leq k$ and $k \leq \operatorname{len} f$. Then $f_k \in \operatorname{Values} G$.

(9) If $n \leq \operatorname{len} f$ and $x \in \widetilde{\mathcal{L}}(f_{\lfloor n})$, then there exists a natural number i such that $n+1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $x \in \mathcal{L}(f,i)$.

(10) If f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $f_k \in \operatorname{left\_cell}(f,k,G)$ and $f_k \in \operatorname{right\_cell}(f,k,G)$.

(11) If f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\operatorname{Int} \operatorname{left\_cell}(f,k,G) \neq \emptyset$ and $\operatorname{Int} \operatorname{right\_cell}(f,k,G) \neq \emptyset$.

(12) Suppose f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f,k,G)$ is connected and $\operatorname{Int} \operatorname{right\_cell}(f,k,G)$ is connected.

(13) If f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\overline{\operatorname{Int} \operatorname{left\_cell}(f,k,G)} = \operatorname{left\_cell}(f,k,G)$ and $\overline{\operatorname{Int} \operatorname{right\_cell}(f,k,G)} = \operatorname{right\_cell}(f,k,G)$.

(14) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f,k)$ is horizontal. Then there exists j such that $1 \leq j$ and $j \leq \operatorname{width} G$ and for every p such that $p \in \mathcal{L}(f,k)$ holds $p_{\mathbf{2}} = (G \circ (1,j))_{\mathbf{2}}$.

(15) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f,k)$ is vertical. Then there exists i such that $1 \leq i$ and $i \leq \operatorname{len} G$ and for every p such that $p \in \mathcal{L}(f,k)$ holds $p_{\mathbf{1}} = (G \circ (i,1))_{\mathbf{1}}$.

(16) If f is a sequence which elements belong to G and special and $i \leq \operatorname{len} G$ and $j \leq \operatorname{width} G$, then $\operatorname{Int} \operatorname{cell}(G,i,j)$ misses $\widetilde{\mathcal{L}}(f)$.

(17) Suppose f is a sequence which elements belong to G and special and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f,k,G)$ misses $\widetilde{\mathcal{L}}(f)$ and $\operatorname{Int} \operatorname{right\_cell}(f,k,G)$ misses $\widetilde{\mathcal{L}}(f)$.

(18) Suppose $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$. Then $(G \circ (i,j))_{\mathbf{1}} = (G \circ (i,j+1))_{\mathbf{1}}$ and $(G \circ (i,j))_{\mathbf{2}} = (G \circ (i+1,j))_{\mathbf{2}}$ and $(G \circ (i+1,j+1))_{\mathbf{1}} = (G \circ (i+1,j))_{\mathbf{1}}$ and $(G \circ (i+1,j+1))_{\mathbf{2}} = (G \circ (i,j+1))_{\mathbf{2}}$.

(19) Let i, j be natural numbers. Suppose $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$. Then $p \in \operatorname{cell}(G,i,j)$ if and only if the following conditions are satisfied:

(i) $(G \circ (i,j))_{\mathbf{1}} \leq p_{\mathbf{1}}$,

(ii) $p_{\mathbf{1}} \leq (G \circ (i+1,j))_{\mathbf{1}}$,

(iii) $(G \circ (i,j))_{\mathbf{2}} \leq p_{\mathbf{2}}$, and

(iv) $p_{\mathbf{2}} \leq (G \circ (i,j+1))_{\mathbf{2}}$.

(20) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\operatorname{cell}(G,i,j) = \{[r,s] : (G \circ (i,j))_{\mathbf{1}} \leq r \wedge r \leq (G \circ (i+1,j))_{\mathbf{1}} \wedge (G \circ (i,j))_{\mathbf{2}} \leq s \wedge s \leq (G \circ (i,j+1))_{\mathbf{2}}\}$.

(21) Suppose $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G,i,j)$. Then $p = G \circ (i,j)$ or $p = G \circ (i,j+1)$ or $p = G \circ (i+1,j+1)$ or $p = G \circ (i+1,j)$.

(22) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $G \circ (i,j) \in \operatorname{cell}(G,i,j)$ and $G \circ (i,j+1) \in \operatorname{cell}(G,i,j)$ and $G \circ (i+1,j+1) \in \operatorname{cell}(G,i,j)$ and $G \circ (i+1,j) \in \operatorname{cell}(G,i,j)$.

(23) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G,i,j)$, then p is extremal in $\operatorname{cell}(G,i,j)$.

(24) Suppose $2 \leq \operatorname{len} G$ and $2 \leq \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Then there exist i, j such that $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$ and $\mathcal{L}(f,k) \subseteq \operatorname{cell}(G,i,j)$.

(25) Suppose $2 \leq \operatorname{len} G$ and $2 \leq \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$ and $p \in \operatorname{Values} G$ and $p \in \mathcal{L}(f,k)$. Then $p = f_k$ or $p = f_{k+1}$.

(26) If $\langle i, j\rangle \in$ the indices of G and $1 \leq k$ and $k \leq \operatorname{width} G$, then $(G \circ (i, j))_{\mathbf{1}} \leq (G \circ (\operatorname{len} G, k))_{\mathbf{1}}$.

(27) If $\langle i, j\rangle \in$ the indices of G and $1 \leq k$ and $k \leq \operatorname{len} G$, then $(G \circ (i, j))_{\mathbf{2}} \leq (G \circ (k, \operatorname{width} G))_{\mathbf{2}}$.

(28) Suppose f is a sequence which elements belong to G and special and $\widetilde{\mathcal{L}}(g) \subseteq \widetilde{\mathcal{L}}(f)$ and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Let A be a subset of $\mathcal{E}^2_{\mathrm{T}}$. If $A = \text{right\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(g)$ or $A = \text{left\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(g)$, then A is connected.

(29) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\text{right\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \text{RightComp}(f)$ and $\text{left\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \text{LeftComp}(f)$.

3. CAGES

In the sequel C denotes a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and i_1, i_2 denote natural numbers.

One can prove the following propositions:

(30) There exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} \text{Gauge}(C,n)$ and $\mathrm{N}_{\min}(C) \in \text{cell}(\text{Gauge}(C,n), i, \operatorname{width} \text{Gauge}(C,n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \text{Gauge}(C,n) \circ (i, \operatorname{width} \text{Gauge}(C,n) -' 1)$.

(31) Suppose that $1 \leq i_1$ and $i_1+1 \leq \operatorname{len} \text{Gauge}(C,n)$ and $\mathrm{N}_{\min}(C) \in \text{cell}(\text{Gauge}(C,n), i_1, \operatorname{width} \text{Gauge}(C,n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \text{Gauge}(C,n) \circ (i_1, \operatorname{width} \text{Gauge}(C,n) -' 1)$ and $1 \leq i_2$ and $i_2+1 \leq \operatorname{len} \text{Gauge}(C,n)$ and $\mathrm{N}_{\min}(C) \in \text{cell}(\text{Gauge}(C,n), i_2, \operatorname{width} \text{Gauge}(C,n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \text{Gauge}(C,n) \circ (i_2, \operatorname{width} \text{Gauge}(C,n) -' 1)$. Then $i_1 = i_2$.

(32) Let f be a standard non constant special circular sequence. Suppose that

(i) f is a sequence which elements belong to $\text{Gauge}(C,n)$,

(ii) for every k such that $1 \leq k$ and $k+1 \leq \operatorname{len} f$ holds $\text{left\_cell}(f,k,\text{Gauge}(C,n))$ misses C and $\text{right\_cell}(f,k,\text{Gauge}(C,n))$ meets C, and

(iii) there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} \text{Gauge}(C,n)$ and $f_1 = \text{Gauge}(C,n) \circ (i, \operatorname{width} \text{Gauge}(C,n))$ and $f_2 = \text{Gauge}(C,n) \circ (i+1, \operatorname{width} \text{Gauge}(C,n))$ and $\mathrm{N}_{\min}(C) \in \text{cell}(\text{Gauge}(C,n), i, \operatorname{width} \text{Gauge}(C,n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \text{Gauge}(C,n) \circ (i, \operatorname{width} \text{Gauge}(C,n) -' 1)$.

Then $\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f)) = f_1$.

Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Let us assume that C is connected. The functor $\text{Cage}(C,n)$ yields a clockwise oriented standard non constant special circular sequence and is defined by the conditions (Def. 1).

(Def. 1)(i) $\text{Cage}(C,n)$ is a sequence which elements belong to $\text{Gauge}(C,n)$,

(ii) there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} \text{Gauge}(C,n)$ and $(\text{Cage}(C,n))_1 = \text{Gauge}(C,n) \circ (i, \operatorname{width} \text{Gauge}(C,n))$ and $(\text{Cage}(C,n))_2 = \text{Gauge}(C,n) \circ (i+1, \operatorname{width} \text{Gauge}(C,n))$ and $\mathrm{N}_{\min}(C) \in \text{cell}(\text{Gauge}(C,n), i, \operatorname{width} \text{Gauge}(C,n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \text{Gauge}(C,n) \circ (i, \operatorname{width} \text{Gauge}(C,n) -' 1)$, and

(iii) for every k such that $1 \leq k$ and $k+2 \leq \operatorname{len} \text{Cage}(C,n)$ holds if $\text{front\_left\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ misses C and $\text{front\_right\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ misses C, then $\text{Cage}(C,n)$ turns right k, $\text{Gauge}(C,n)$ and if $\text{front\_left\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ misses C and $\text{front\_right\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ meets C, then $\text{Cage}(C,n)$ goes straight k, $\text{Gauge}(C,n)$ and if $\text{front\_left\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ meets C, then $\text{Cage}(C,n)$ turns left k, $\text{Gauge}(C,n)$.

Next we state two propositions:

(33) If C is connected and $1 \leq k$ and $k+1 \leq \operatorname{len} \text{Cage}(C,n)$, then $\text{left\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ misses C and $\text{right\_cell}(\text{Cage}(C,n), k, \text{Gauge}(C,n))$ meets C.

(34) If C is connected, then $\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(\text{Cage}(C,n))) = (\text{Cage}(C,n))_1$.

ACKNOWLEDGMENTS

Thanks are due to Professor Yatsuka Nakamura and people in the Kiso Laboratory for their gracious support during the work on this paper. Authors would like also to express their gratitude to Andrzej Trybulec for his help in preparing the paper.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[5] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/pcomps_1.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[9] Czesław Byliński. Some properties of restrictions of finite sequences. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/finseq_5.html.

[10] Czesław Byliński. Gauges. *Journal of Formalized Mathematics*, 11, 1999. http://mizar.org/JFM/Vol11/jordan8.html.

[11] Czesław Byliński. Some properties of cells on go board. *Journal of Formalized Mathematics*, 11, 1999. http://mizar.org/JFM/Vol11/gobrd13.html.

[12] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/pscomp_1.html.

[13] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/compts_1.html.

[14] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[15] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_T$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[16] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/matrix_1.html.

[17] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/metric_1.html.

[18] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfinseq.html.

[19] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-Board — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/goboard1.html.

[20] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons, part I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/sppol_1.html.

[21] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-Board into cells. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/goboard5.html.

[22] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/binarith.html.

[23] Beata Padlewska. Connected spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/connsp_1.html.

[24] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[26] Andrzej Trybulec. Left and right component of the complement of a special closed curve. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/goboard9.html.

[27] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[28] Andrzej Trybulec and Yatsuka Nakamura. On the order on a special polygon. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/sprect_2.html.

[29] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[30] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[31] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[32] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_1.html.

Received June 22, 1999

Published January 2, 2004