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# On the Dividing Function of the Simple Closed Curve into Segments

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**Summary.** At the beginning, the concept of the segment of the simple closed curve in 2-dimensional Euclidean space is defined. Some properties of segments are shown in the succeeding theorems. At the end, the existence of the function which can divide the simple closed curve into segments is shown. We can make the diameter of segments as small as we want.

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The articles [19], [22], [20], [1], [23], [17], [2], [3], [4], [21], [10], [11], [12], [14], [15], [18], [7], [6], [8], [5], [13], [16], and [9] provide the notation and terminology for this paper.

## 1. Definition of the Segment and Its Property

In this paper $p$, $p_1$, $q$ denote points of $\mathcal{E}^2_{\rm T}$.

The following three propositions are true:

(1) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P$ is a simple closed curve. Then $\mathrm{W}_{\rm min}(P) \in \mathrm{LowerArc}(P)$ and $\mathrm{E}_{\rm max}(P) \in \mathrm{LowerArc}(P)$ and $\mathrm{W}_{\rm min}(P) \in \mathrm{UpperArc}(P)$ and $\mathrm{E}_{\rm max}(P) \in \mathrm{UpperArc}(P)$.

(2) For every compact non empty subset $P$ of $\mathcal{E}^2_{\rm T}$ and for every $q$ such that $P$ is a simple closed curve and $q \leq_P \mathrm{W}_{\rm min}(P)$ holds $q = \mathrm{W}_{\rm min}(P)$.

(3) For every compact non empty subset $P$ of $\mathcal{E}^2_{\rm T}$ and for every $q$ such that $P$ is a simple closed curve and $q \in P$ holds $\mathrm{W}_{\rm min}(P) \leq_P q$.

Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and let $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{Segment}(q_1, q_2, P)$ yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 1) $$\mathrm{Segment}(q_1,q_2,P) = \begin{cases} \{p : q_1 \leq_P p \land p \leq_P q_2\}, & \text{if } q_2 \neq \mathrm{W}_{\rm min}(P), \\ \{p_1 : q_1 \leq_P p_1 \lor q_1 \in P \land p_1 = \mathrm{W}_{\rm min}(P)\}, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(4) For every compact non empty subset $P$ of $\mathcal{E}^2_{\rm T}$ such that $P$ is a simple closed curve holds $\mathrm{Segment}(\mathrm{W}_{\rm min}(P), \mathrm{E}_{\rm max}(P), P) = \mathrm{UpperArc}(P)$ and $\mathrm{Segment}(\mathrm{E}_{\rm max}(P), \mathrm{W}_{\rm min}(P), P) = \mathrm{LowerArc}(P)$.

(5) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. If $P$ is a simple closed curve and $q_1 \leq_P q_2$, then $q_1 \in P$ and $q_2 \in P$.

(6) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. If $P$ is a simple closed curve and $q_1 \leq_P q_2$, then $q_1 \in \mathrm{Segment}(q_1, q_2, P)$ and $q_2 \in \mathrm{Segment}(q_1, q_2, P)$.

(7) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$ be a point of $\mathcal{E}^2_{\rm T}$. If $q_1 \in P$ and $P$ is a simple closed curve, then $q_1 \in \mathrm{Segment}(q_1, \mathrm{W_{min}}(P), P)$.

(8) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q$ be a point of $\mathcal{E}^2_{\rm T}$. If $P$ is a simple closed curve and $q \in P$ and $q \neq \mathrm{W_{min}}(P)$, then $\mathrm{Segment}(q, q, P) = \{q\}$.

(9) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. If $P$ is a simple closed curve and $q_1 \neq \mathrm{W_{min}}(P)$ and $q_2 \neq \mathrm{W_{min}}(P)$, then $\mathrm{W_{min}}(P) \notin \mathrm{Segment}(q_1, q_2, P)$.

(10) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$, $q_3$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $P$ is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_1 = q_2$ and $q_1 = \mathrm{W_{min}}(P)$ and $q_1 \neq q_3$ and $q_2 = q_3$ and $q_2 = \mathrm{W_{min}}(P)$. Then $\mathrm{Segment}(q_1, q_2, P) \cap \mathrm{Segment}(q_2, q_3, P) = \{q_2\}$.

(11) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $P$ is a simple closed curve and $q_1 \leq_P q_2$ and $q_1 \neq \mathrm{W_{min}}(P)$ and $q_2 \neq \mathrm{W_{min}}(P)$. Then $\mathrm{Segment}(q_1, q_2, P) \cap \mathrm{Segment}(q_2, \mathrm{W_{min}}(P), P) = \{q_2\}$.

(12) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $P$ is a simple closed curve and $q_1 \leq_P q_2$ and $q_1 \neq q_2$ and $q_1 \neq \mathrm{W_{min}}(P)$. Then $\mathrm{Segment}(q_2, \mathrm{W_{min}}(P), P) \cap \mathrm{Segment}(\mathrm{W_{min}}(P), q_1, P) = \{\mathrm{W_{min}}(P)\}$.

(13) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$, $q_3$, $q_4$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $P$ is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$ and $q_1 \neq q_2$ and $q_2 \neq q_3$. Then $\mathrm{Segment}(q_1, q_2, P)$ misses $\mathrm{Segment}(q_3, q_4, P)$.

(14) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and $q_1$, $q_2$, $q_3$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $P$ is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_1 \neq \mathrm{W_{min}}(P)$ and $q_1 \neq q_2$ and $q_2 \neq q_3$. Then $\mathrm{Segment}(q_1, q_2, P)$ misses $\mathrm{Segment}(q_3, \mathrm{W_{min}}(P), P)$.

## 2. A Function to Divide the Simple Closed Curve

In the sequel $n$ is a natural number.

One can prove the following propositions:

(15) Let $P$ be a non empty subset of $\mathcal{E}^n_{\rm T}$ and $f$ be a map from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$. Suppose $P \neq \emptyset$ and $f$ is a homeomorphism. Then there exists a map $g$ from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ such that $f = g$ and $g$ is continuous and one-to-one.

(16) Let $P$ be a non empty subset of $\mathcal{E}^n_{\rm T}$ and $g$ be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$. Suppose $g$ is continuous and one-to-one and $\mathrm{rng}\, g = P$. Then there exists a map $f$ from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$ such that $f = g$ and $f$ is a homeomorphism.

One can verify that every finite sequence of elements of $\mathbb{R}$ which is increasing is also one-to-one.

Next we state several propositions:

(17) For every finite sequence $f$ of elements of $\mathbb{R}$ such that $f$ is increasing holds $f$ is one-to-one.

(18) Let $A$ be a subset of $\mathcal{E}^2_{\rm T}$ and $p_1$, $p_2$ be points of $\mathcal{E}^2_{\rm T}$. Suppose $A$ is an arc from $p_1$ to $p_2$. Then there exists a map $g$ from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$ such that $g$ is continuous and one-to-one and $\mathrm{rng}\, g = A$ and $g(0) = p_1$ and $g(1) = p_2$.

(19) Let $P$ be a non empty subset of $\mathcal{E}^2_{\rm T}$, $p_1$, $p_2$, $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$, $g$ be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and $s_1$, $s_2$ be real numbers. Suppose that $P$ is an arc from $p_1$ to $p_2$ and $g$ is continuous and one-to-one and $\mathrm{rng}\, g = P$ and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leq s_1$ and $s_1 \leq 1$ and $g(s_2) = q_2$ and $0 \leq s_2$ and $s_2 \leq 1$ and $s_1 \leq s_2$. Then LE $q_1$, $q_2$, $P$, $p_1$, $p_2$.

(20) Let $P$ be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, $p_1$, $p_2$, $q_1$, $q_2$ be points of $\mathcal{E}^2_{\mathrm{T}}$, $g$ be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and $s_1$, $s_2$ be real numbers. Suppose that $g$ is continuous and one-to-one and $\operatorname{rng} g = P$ and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leq s_1$ and $s_1 \leq 1$ and $g(s_2) = q_2$ and $0 \leq s_2$ and $s_2 \leq 1$ and LE $q_1$, $q_2$, $P$, $p_1$, $p_2$. Then $s_1 \leq s_2$.

(21) Let $P$ be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and $e$ be a real number. Suppose $P$ is a simple closed curve and $e > 0$. Then there exists a finite sequence $h$ of elements of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that

$h(1) = \mathrm{W}_{\min}(P)$ and $h$ is one-to-one and $8 \leq \operatorname{len} h$ and $\operatorname{rng} h \subseteq P$ and for every natural number $i$ such that $1 \leq i$ and $i < \operatorname{len} h$ holds $h_i \leq_P h_{i+1}$ and for every natural number $i$ and for every subset $W$ of $\mathcal{E}^2$ such that $1 \leq i$ and $i < \operatorname{len} h$ and $W = \operatorname{Segment}(h_i, h_{i+1}, P)$ holds $\varnothing W < e$ and for every subset $W$ of $\mathcal{E}^2$ such that $W = \operatorname{Segment}(h_{\operatorname{len} h}, h_1, P)$ holds $\varnothing W < e$ and for every natural number $i$ such that $1 \leq i$ and $i + 1 < \operatorname{len} h$ holds $\operatorname{Segment}(h_i, h_{i+1}, P) \cap \operatorname{Segment}(h_{i+1}, h_{i+2}, P) = \{h_{i+1}\}$ and $\operatorname{Segment}(h_{\operatorname{len} h}, h_1, P) \cap \operatorname{Segment}(h_1, h_2, P) = \{h_1\}$ and $\operatorname{Segment}(h_{\operatorname{len} h -' 1}, h_{\operatorname{len} h}, P) \cap \operatorname{Segment}(h_{\operatorname{len} h}, h_1, P) = \{h_{\operatorname{len} h}\}$ and $\operatorname{Segment}(h_{\operatorname{len} h -' 1}, h_{\operatorname{len} h}, P)$ misses $\operatorname{Segment}(h_1, h_2, P)$ and for all natural numbers $i$, $j$ such that $1 \leq i$ and $i < j$ and $j < \operatorname{len} h$ and $i$ and $j$ are not adjacent holds $\operatorname{Segment}(h_i, h_{i+1}, P)$ misses $\operatorname{Segment}(h_j, h_{j+1}, P)$ and for every natural number $i$ such that $1 < i$ and $i + 1 < \operatorname{len} h$ holds $\operatorname{Segment}(h_{\operatorname{len} h}, h_1, P)$ misses $\operatorname{Segment}(h_i, h_{i+1}, P)$.

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