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A Decomposition of Simple Closed Curves and the Order of Their Points

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Summary. The goal of the article is to introduce an order on a simple closed curve. To do this, we fix two points on the curve and devide it into two arcs. We prove that such a decomposition is unique. Other auxiliary theorems about arcs are proven for preparation of the proof of the above.

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The articles [18], [21], [1], [20], [12], [17], [22], [3], [4], [9], [10], [15], [8], [16], [6], [19], [7], [13], [2], [14], [11], and [5] provide the notation and terminology for this paper.

1. Middle Points of Arcs

We follow the rules: s, r denote real numbers, n denotes a natural number, and p, q denote points of $\mathcal{E}^2_{\rm T}$.

We now state a number of propositions:

(2)[1] If $r \leq s$, then $r \leq \frac{r+s}{2}$ and $\frac{r+s}{2} \leq s$.

(3) Let T_1 be a non empty topological space, P be a subset of T_1, A be a subset of $T_1 \restriction P$, and B be a subset of T_1. If B is closed and $A = B \cap P$, then A is closed.

(4) Let T_1, T_2 be non empty topological spaces, P be a non empty subset of T_2, and f be a map from T_1 into $T_2 \restriction P$. Then

(i) f is a map from T_1 into T_2, and

(ii) for every map f_2 from T_1 into T_2 such that $f_2 = f$ and f is continuous holds f_2 is continuous.

(5) For every real number r and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} \geq r\}$ holds P is closed.

(6) For every real number r and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} \leq r\}$ holds P is closed.

(7) For every real number r and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} = r\}$ holds P is closed.

[1] The proposition (1) has been removed.

(8) For every real number r and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \geq r\}$ holds P is closed.

(9) For every real number r and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} \leq r\}$ holds P is closed.

(10) For every real number r and for every subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{2}} = r\}$ holds P is closed.

(11) For every subset P of $\mathcal{E}^n_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds P is connected.

(12) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds P is closed.

(13) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then there exists a point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in P$ and $q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}}{2}$.

(14) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

(15) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor xMiddle$(P, p_1, p_2)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 1) For every subset Q of $\mathcal{E}^2_{\mathrm{T}}$ such that $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}}{2}\}$ holds xMiddle$(P, p_1, p_2) =$ FPoint(P, p_1, p_2, Q).

Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor yMiddle$(P, p_1, p_2)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 2) For every subset Q of $\mathcal{E}^2_{\mathrm{T}}$ such that $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}}{2}\}$ holds yMiddle$(P, p_1, p_2) =$ FPoint(P, p_1, p_2, Q).

One can prove the following three propositions:

(16) Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then xMiddle$(P, p_1, p_2) \in P$ and yMiddle$(P, p_1, p_2) \in P$.

(17) Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $p_1 =$ xMiddle(P, p_1, p_2) iff $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$.

(18) Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $p_1 =$ yMiddle(P, p_1, p_2) iff $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$.

2. Segments of Arcs

The following proposition is true

(19) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2, then LE q_2, q_1, P, p_2, p_1.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor LSegment$(P, p_1, p_2, q_1)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 3) LSegment$(P, p_1, p_2, q_1) = \{q :$ LE q, q_1, P, p_1, $p_2\}$.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor RSegment$(P, p_1, p_2, q_1)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 4) $\mathrm{RSegment}(P, p_1, p_2, q_1) = \{q : \mathrm{LE}\ q_1, q, P, p_1, p_2\}$.

The following propositions are true:

(20) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LSegment}(P, p_1, p_2, q_1) \subseteq P$.

(21) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{RSegment}(P, p_1, p_2, q_1) \subseteq P$.

(22) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{LSegment}(P, p_1, p_2, p_1) = \{p_1\}$.

(25)[2] For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{LSegment}(P, p_1, p_2, p_2) = P$.

(26) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{RSegment}(P, p_1, p_2, p_2) = \{p_2\}$.

(27) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $\mathrm{RSegment}(P, p_1, p_2, p_1) = P$.

(28) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then $\mathrm{RSegment}(P, p_1, p_2, q_1) = \mathrm{LSegment}(P, p_2, p_1, q_1)$.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 5) $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{RSegment}(P, p_1, p_2, q_1) \cap \mathrm{LSegment}(P, p_1, p_2, q_2)$.

One can prove the following four propositions:

(29) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \{q : \mathrm{LE}\ q_1, q, P, p_1, p_2 \wedge \mathrm{LE}\ q, q_2, P, p_1, p_2\}$.

(30) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then $\mathrm{LE}\ q_1, q_2, P, p_1, p_2$ if and only if $\mathrm{LE}\ q_2, q_1, P, p_2, p_1$.

(31) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q \in P$, then $\mathrm{LSegment}(P, p_1, p_2, q) = \mathrm{RSegment}(P, p_2, p_1, q)$.

(32) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$, then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{Segment}(P, p_2, p_1, q_2, q_1)$.

3. Decomposition of a Simple Closed Curve Into Two Arcs

Let s be a real number. The functor $\mathrm{VerticalLine}(s)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 6) $\mathrm{VerticalLine}(s) = \{p;\ p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} = s\}$.

The functor $\mathrm{HorizontalLine}(s)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 7) $\mathrm{HorizontalLine}(s) = \{p : p_{\mathbf{2}} = s\}$.

We now state four propositions:

(33) For every real number r holds $\mathrm{VerticalLine}(r)$ is closed and $\mathrm{HorizontalLine}(r)$ is closed.

(34) For every real number r and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p \in \mathrm{VerticalLine}(r)$ iff $p_{\mathbf{1}} = r$.

(35) For every real number r and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p \in \mathrm{HorizontalLine}(r)$ iff $p_{\mathbf{2}} = r$.

[2] The propositions (23) and (24) have been removed.

(40)[3] Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve. Then there exist non empty subsets P_1, P_2 of $\mathcal{E}^2_{\rm T}$ such that

(i) P_1 is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$,

(ii) P_2 is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$,

(iii) $P_1 \cap P_2 = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$,

(iv) $P_1 \cup P_2 = P$, and

(v) $(\mathrm{FPoint}(P_1, \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(P_2, \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}$

4. UNIQUENESS OF DECOMPOSITION OF A SIMPLE CLOSED CURVE

The following propositions are true:

(41) For every subset P of $\mathbb{I}$ such that $P = (\text{the carrier of } \mathbb{I}) \setminus \{0, 1\}$ holds P is open.

(44)[4] For all real numbers r, s holds $]r, s[$ misses $\{r, s\}$.

(45) For all real numbers a, b, c holds $c \in\,]a, b[$ iff $a < c$ and $c < b$.

(46) For every subset P of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers r, s such that $P =]r, s[$ holds P is open.

(47) Let S be a non empty topological space, P_1, P_2 be subsets of S, and P_1' be a subset of $S{\restriction}P_2$. If $P_1 = P_1'$ and $P_1 \subseteq P_2$, then $S{\restriction}P_1 = S{\restriction}P_2{\restriction}P_1'$.

(48) For every subset P_7 of $\mathbb{I}$ such that $P_7 = (\text{the carrier of } \mathbb{I}) \setminus \{0, 1\}$ holds $P_7 \neq \emptyset$ and P_7 is connected.

(49) For every subset P of $\mathcal{E}^n_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that P is an arc from p_1 to p_2 holds $p_1 \neq p_2$.

(50) Let P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}){\restriction}P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is open.

(52)[5] Let P be a subset of $\mathcal{E}^n_{\rm T}$, P_1, P_2 be non empty subsets of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}){\restriction}P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose $p_1 \in P$ and $p_2 \in P$ and P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \{p_1, p_2\}$ and $Q = P_1 \setminus \{p_1, p_2\}$. Then Q is open.

(53) Let P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}){\restriction}P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is connected.

(54) Let G_1 be a non empty topological space, P_1, P be subsets of G_1, Q' be a subset of $G_1{\restriction}P_1$, and Q be a subset of $G_1{\restriction}P$. If $P_1 \subseteq P$ and $Q = Q'$ and Q' is connected, then Q is connected.

(55) Let P be a subset of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a point p_3 of $\mathcal{E}^n_{\rm T}$ such that $p_3 \in P$ and $p_3 \neq p_1$ and $p_3 \neq p_2$.

(56) For every subset P of $\mathcal{E}^n_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that P is an arc from p_1 to p_2 holds $P \setminus \{p_1, p_2\} \neq \emptyset$.

(57) Let P_1 be a subset of $\mathcal{E}^n_{\rm T}$, P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a subset of $(\mathcal{E}^n_{\rm T}){\restriction}P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P_1 is an arc from p_1 to p_2 and $P_1 \subseteq P$ and $Q = P_1 \setminus \{p_1, p_2\}$, then Q is connected.

(58) Let T, S, V be non empty topological spaces, P_1 be a non empty subset of S, P_2 be a subset of S, f be a map from T into $S{\restriction}P_1$, and g be a map from $S{\restriction}P_2$ into V. Suppose $P_1 \subseteq P_2$ and f is continuous and g is continuous. Then there exists a map h from T into V such that $h = g \cdot f$ and h is continuous.

[3] The propositions (36)–(39) have been removed.

[4] The propositions (42) and (43) have been removed.

[5] The proposition (51) has been removed.

(59) Let P_1, P_2 be subsets of $\mathcal{E}^n_\mathrm{T}$ and p_1, p_2 be points of $\mathcal{E}^n_\mathrm{T}$. If P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(60) Let P be a non empty subset of $\mathcal{E}^2_\mathrm{T}$, Q be a subset of $(\mathcal{E}^2_\mathrm{T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^2_\mathrm{T}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_1 \neq p_2$ and $Q = P \setminus \{p_1, p_2\}$. Then Q is not connected.

(61) Let P be a non empty subset of $\mathcal{E}^2_\mathrm{T}$, P_1, P_2, P'_1, P'_2 be subsets of $\mathcal{E}^2_\mathrm{T}$, and p_1, p_2 be points of $\mathcal{E}^2_\mathrm{T}$. Suppose that P is a simple closed curve and P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \cup P_2 = P$ and P'_1 is an arc from p_1 to p_2 and P'_2 is an arc from p_1 to p_2 and $P'_1 \cup P'_2 = P$. Then $P_1 = P'_1$ and $P_2 = P'_2$ or $P_1 = P'_2$ and $P_2 = P'_1$.

5. Lower Arcs and Upper Arcs

Let us observe that every element of $\mathbb{R}^{\mathbf{1}}$ is real.

One can prove the following proposition

(64)[6] Let P_1 be a subset of $\mathcal{E}^2_\mathrm{T}$, r be a real number, and p_1, p_2 be points of $\mathcal{E}^2_\mathrm{T}$. Suppose $(p_1)_{\mathbf{1}} \leq r$ and $r \leq (p_2)_{\mathbf{1}}$ and P_1 is an arc from p_1 to p_2. Then P_1 meets $\mathrm{VerticalLine}(r)$ and $P_1 \cap \mathrm{VerticalLine}(r)$ is closed.

Let P be a compact non empty subset of $\mathcal{E}^2_\mathrm{T}$. Let us assume that P is a simple closed curve. The functor $\mathrm{UpperArc}(P)$ yielding a non empty subset of $\mathcal{E}^2_\mathrm{T}$ is defined by the conditions (Def. 8).

(Def. 8)(i) $\mathrm{UpperArc}(P)$ is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$, and

(ii) there exists a non empty subset P_2 of $\mathcal{E}^2_\mathrm{T}$ such that P_2 is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and $\mathrm{UpperArc}(P) \cap P_2 = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup P_2 = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}(P), \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(P_2, \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

Let P be a compact non empty subset of $\mathcal{E}^2_\mathrm{T}$. Let us assume that P is a simple closed curve. The functor $\mathrm{LowerArc}(P)$ yielding a non empty subset of $\mathcal{E}^2_\mathrm{T}$ is defined as follows:

(Def. 9) $\mathrm{LowerArc}(P)$ is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and $\mathrm{UpperArc}(P) \cap \mathrm{LowerArc}(P) = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup \mathrm{LowerArc}(P) = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}(P), \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}($ $(\mathrm{LPoint}(\mathrm{LowerArc}(P), \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

The following propositions are true:

(65) Let P be a compact non empty subset of $\mathcal{E}^2_\mathrm{T}$. Suppose P is a simple closed curve. Then $\mathrm{UpperArc}(P)$ is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$ and $\mathrm{UpperArc}(P)$ is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and $\mathrm{LowerArc}(P)$ is an arc from $\mathrm{E_{max}}(P)$ to $\mathrm{W_{min}}(P)$ and $\mathrm{LowerArc}(P)$ is an arc from $\mathrm{W_{min}}(P)$ to $\mathrm{E_{max}}(P)$ and $\mathrm{UpperArc}(P) \cap \mathrm{LowerArc}(P) = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup \mathrm{LowerArc}(P) = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}(P), \mathrm{W_{min}}(P), \mathrm{E_{max}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}} > (\mathrm{LPoint}(\mathrm{LowerArc}(P), \mathrm{E_{max}}(P), \mathrm{W_{min}}(P), \mathrm{VerticalLine}(\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2})))_{\mathbf{2}}$.

(66) Let P be a compact non empty subset of $\mathcal{E}^2_\mathrm{T}$. If P is a simple closed curve, then $\mathrm{LowerArc}(P) = (P \setminus \mathrm{UpperArc}(P)) \cup \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) = (P \setminus \mathrm{LowerArc}(P)) \cup \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$.

(67) Let P be a compact non empty subset of $\mathcal{E}^2_\mathrm{T}$ and P_1 be a subset of $(\mathcal{E}^2_\mathrm{T}) \restriction P$. If P is a simple closed curve and $\mathrm{UpperArc}(P) \cap P_1 = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $\mathrm{UpperArc}(P) \cup P_1 = P$, then $P_1 = \mathrm{LowerArc}(P)$.

(68) Let P be a compact non empty subset of $\mathcal{E}^2_\mathrm{T}$ and P_1 be a subset of $(\mathcal{E}^2_\mathrm{T}) \restriction P$. If P is a simple closed curve and $P_1 \cap \mathrm{LowerArc}(P) = \{\mathrm{W_{min}}(P), \mathrm{E_{max}}(P)\}$ and $P_1 \cup \mathrm{LowerArc}(P) = P$, then $P_1 = \mathrm{UpperArc}(P)$.

[6] The propositions (62) and (63) have been removed.

6. An Order of Points in a Simple Closed Curve

We now state two propositions:

(69) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE q, p_1, P, p_1, p_2, then $q = p_1$.

(70) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE p_2, q, P, p_1, p_2, then $q = p_2$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and let q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. The predicate $q_1 \leq_P q_2$ is defined by the conditions (Def. 10).

(Def. 10)(i) $q_1 \in \mathrm{UpperArc}(P)$ and $q_2 \in \mathrm{LowerArc}(P)$ and $q_2 \neq \mathrm{W_{min}}(P)$, or

(ii) $q_1 \in \mathrm{UpperArc}(P)$ and $q_2 \in \mathrm{UpperArc}(P)$ and LE q_1, q_2, $\mathrm{UpperArc}(P)$, $\mathrm{W_{min}}(P)$, $\mathrm{E_{max}}(P)$, or

(iii) $q_1 \in \mathrm{LowerArc}(P)$ and $q_2 \in \mathrm{LowerArc}(P)$ and $q_2 \neq \mathrm{W_{min}}(P)$ and LE q_1, q_2, $\mathrm{LowerArc}(P)$, $\mathrm{E_{max}}(P)$, $\mathrm{W_{min}}(P)$.

One can prove the following propositions:

(71) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and q be a point of $\mathcal{E}^2_{\rm T}$. If P is a simple closed curve and $q \in P$, then $q \leq_P q$.

(72) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. If P is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_1$, then $q_1 = q_2$.

(73) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$ and q_1, q_2, q_3 be points of $\mathcal{E}^2_{\rm T}$. If P is a simple closed curve and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$, then $q_1 \leq_P q_3$.

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