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The Ordering of Points on a Curve. Part II

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Summary. The proof of the Jordan Curve Theorem according to [11] is continued. The notions of the first and last point of a oriented arc are introduced as well as ordering of points on a curve in $\mathcal{E}^2_T$.

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The articles [12], [14], [1], [15], [2], [3], [4], [7], [13], [9], [8], [10], [5], and [6] provide the notation and terminology for this paper.

1. First and last point of a curve

One can prove the following proposition

(1) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$, and s_1 be a real number. Suppose that P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_1 \in Q$ and $f(s_1) = q_1$ and f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$ and $0 \leq s_1$ and $s_1 \leq 1$ and for every real number t such that $0 \leq t$ and $t < s_1$ holds $f(t) \notin Q$. Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$ and s_2 be a real number. Suppose g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = q_1$ and $0 \leq s_2$ and $s_2 \leq 1$. Let t be a real number. If $0 \leq t$ and $t < s_2$, then $g(t) \notin Q$.

Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. The functor $\mathrm{FPoint}(P, p_1, p_2, Q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{FPoint}(P, p_1, p_2, Q) \in P \cap Q$, and

(ii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$ and for every real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = \mathrm{FPoint}(P, p_1, p_2, Q)$ and $0 \leq s_2$ and $s_2 \leq 1$ and for every real number t such that $0 \leq t$ and $t < s_2$ holds $g(t) \notin Q$.

The following propositions are true:

(2) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in P$ and P is an arc from p_1 to p_2 and $Q = \{p\}$, then $\mathrm{FPoint}(P, p_1, p_2, Q) = p$.

(3) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 \in Q$ and $P \cap Q$ is closed and P is an arc from p_1 to p_2, then $\mathrm{FPoint}(P, p_1, p_2, Q) = p_1$.

(4) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s_1 be a real number. Suppose that P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_1 \in Q$ and $f(s_1) = q_1$ and f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$ and $0 \leq s_1$ and $s_1 \leq 1$ and for every real number t such that $1 \geq t$ and $t > s_1$ holds $f(t) \notin Q$. Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$ and s_2 be a real number. Suppose g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = q_1$ and $0 \leq s_2$ and $s_2 \leq 1$. Let t be a real number. If $1 \geq t$ and $t > s_2$, then $g(t) \notin Q$.

Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Let us assume that P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. The functor $\mathrm{LPoint}(P, p_1, p_2, Q)$ yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{LPoint}(P, p_1, p_2, Q) \in P \cap Q$, and

(ii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$ and for every real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = \mathrm{LPoint}(P, p_1, p_2, Q)$ and $0 \leq s_2$ and $s_2 \leq 1$ and for every real number t such that $1 \geq t$ and $t > s_2$ holds $g(t) \notin Q$.

One can prove the following three propositions:

(5) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p, p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p \in P$ and P is an arc from p_1 to p_2 and $Q = \{p\}$, then $\mathrm{LPoint}(P, p_1, p_2, Q) = p$.

(6) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_2 \in Q$ and $P \cap Q$ is closed and P is an arc from p_1 to p_2, then $\mathrm{LPoint}(P, p_1, p_2, Q) = p_2$.

(7) Let P be a subset of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $P \subseteq Q$ and P is closed and an arc from p_1 to p_2. Then $\mathrm{FPoint}(P, p_1, p_2, Q) = p_1$ and $\mathrm{LPoint}(P, p_1, p_2, Q) = p_2$.

2. The Ordering of Points on a Curve

Let P be a subset of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. We say that LE q_1, q_2, P, p_1, p_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $q_1 \in P$,

(ii) $q_2 \in P$, and

(iii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$ and for all real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leq s_1$ and $s_1 \leq 1$ and $g(s_2) = q_2$ and $0 \leq s_2$ and $s_2 \leq 1$ holds $s_1 \leq s_2$.

Next we state several propositions:

(8) Let P be a subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$, g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s_1, s_2 be real numbers. Suppose that P is an arc from p_1 to p_2 and g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leq s_1$ and $s_1 \leq 1$ and $g(s_2) = q_2$ and $0 \leq s_2$ and $s_2 \leq 1$ and $s_1 \leq s_2$. Then LE q_1, q_2, P, p_1, p_2.

(9) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then LE q_1, q_1, P, p_1, p_2.

(10) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$. Then LE p_1, q_1, P, p_1, p_2 and LE q_1, p_2, P, p_1, p_2.

(11) For every subset P of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that P is an arc from p_1 to p_2 holds LE p_1, p_2, P, p_1, p_2.

(12) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_1, P, p_1, p_2. Then $q_1 = q_2$.

(13) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2, q_3 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_3, P, p_1, p_2. Then LE q_1, q_3, P, p_1, p_2.

(14) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq q_2$. Then LE q_1, q_2, P, p_1, p_2 and not LE q_2, q_1, P, p_1, p_2 or LE q_2, q_1, P, p_1, p_2 and not LE q_1, q_2, P, p_1, p_2.

3. Some properties of the ordering of points on a curve

We now state a number of propositions:

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $\widetilde{\mathcal{L}}(f) \cap Q$ is closed and $q \in \widetilde{\mathcal{L}}(f)$ and $q \in Q$. Then LE FPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q)$, q, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(16) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $\widetilde{\mathcal{L}}(f) \cap Q$ is closed and $q \in \widetilde{\mathcal{L}}(f)$ and $q \in Q$. Then LE q, LPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q)$, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(17) For all points q_1, q_2, p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that $p_1 \neq p_2$ holds if LE q_1, q_2, $\mathcal{L}(p_1, p_2)$, p_1, p_2, then $q_1 \leq_{p_1,p_2} q_2$.

(18) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and P meets Q and $P \cap Q$ is closed. Then FPoint$(P, p_1, p_2, Q) =$ LPoint(P, p_2, p_1, Q) and LPoint$(P, p_1, p_2, Q) =$ FPoint(P, p_2, p_1, Q).

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and Q is closed and f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and FPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f, i)$. Then FPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) =$ FPoint$(\mathcal{L}(f, i), f_i, f_{i+1}, Q)$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and Q is closed and f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and LPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f, i)$. Then LPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) =$ LPoint$(\mathcal{L}(f, i), f_i, f_{i+1}, Q)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i be a natural number. Suppose $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and f is a special sequence and FPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) \in \mathcal{L}(f, i)$. Then FPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) = f_i$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i be a natural number. Suppose $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and f is a special sequence and LPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) \in \mathcal{L}(f, i)$. Then LPoint$(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) = f_{i+1}$.

(23) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i be a natural number. Suppose f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$. Then LE f_i, f_{i+1}, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i, j be natural numbers. Suppose f is a special sequence and $1 \leq i$ and $i \leq j$ and $j \leq \mathrm{len}\, f$. Then LE f_i, f_j, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(25) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, q be a point of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f, i)$. Then LE f_i, q, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(26) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, q be a point of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f, i)$. Then LE q, f_{i+1}, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(27) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, q be a point of $\mathcal{E}^2_{\rm T}$, and i, j be natural numbers. Suppose that $\widetilde{\mathcal{L}}(f)$ meets Q and f is a special sequence and Q is closed and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f,i)$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f,j)$ and $1 \leq j$ and $j+1 \leq \mathrm{len}\, f$ and $q \in Q$ and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \neq q$. Then $i \leq j$ and if $i = j$, then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \leq_{f_i, f_{i+1}} q$.

(28) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, q be a point of $\mathcal{E}^2_{\rm T}$, and i, j be natural numbers. Suppose that $\widetilde{\mathcal{L}}(f)$ meets Q and f is a special sequence and Q is closed and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f,i)$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f,j)$ and $1 \leq j$ and $j+1 \leq \mathrm{len}\, f$ and $q \in Q$ and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \neq q$. Then $i \geq j$ and if $i = j$, then $q \leq_{f_i, f_{i+1}} \mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q)$.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose $q_1 \in \mathcal{L}(f,i)$ and $q_2 \in \mathcal{L}(f,i)$ and f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$. If LE q_1, q_2, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$, then LE q_1, q_2, $\mathcal{L}(f,i)$, f_i, f_{i+1}.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $q_1 \in \widetilde{\mathcal{L}}(f)$ and $q_2 \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $q_1 \neq q_2$. Then LE q_1, q_2, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$ if and only if for all natural numbers i, j such that $q_1 \in \mathcal{L}(f,i)$ and $q_2 \in \mathcal{L}(f,j)$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $1 \leq j$ and $j+1 \leq \mathrm{len}\, f$ holds $i \leq j$ and if $i = j$, then $q_1 \leq_{f_i, f_{i+1}} q_2$.

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