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Reconstructions of Special Sequences[1]

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Summary. We discuss here some methods for reconstructing special sequences which generate special polygonal arcs in $\mathcal{E}^2_{\mathrm{T}}$. For such reconstructions we introduce a “mid” function which cuts out the middle part of a sequence; the “⇃” function, which cuts down the left part of a sequence at some point; the “⇂” function for cutting down the right part at some point; and the “⇃⇂” function for cutting down both sides at two given points.

We also introduce some methods glueing two special sequences. By such cutting and glueing methods, the speciality of sequences (generatability of special polygonal arcs) is shown to be preserved.

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The articles [14], [17], [2], [3], [15], [10], [1], [11], [12], [18], [5], [4], [16], [6], [9], [8], [13], and [7] provide the notation and terminology for this paper.

1. PRELIMINARIES

In this paper i, i_1, i_2, n denote natural numbers.

Next we state a number of propositions:

(1) If $i -' i_1 \geq 1$ or $i - i_1 \geq 1$, then $i -' i_1 = i - i_1$.

(2) $n -' 0 = n$.

(3) $i_1 - i_2 \leq i_1 -' i_2$.

(4) If $i_1 \leq i_2$, then $n -' i_2 \leq n -' i_1$.

(5) If $i_1 \leq i_2$, then $i_1 -' n \leq i_2 -' n$.

(6) If $i -' i_1 \geq 1$ or $i - i_1 \geq 1$, then $(i -' i_1) + i_1 = i$.

(7) If $i_1 \leq i_2$, then $i_1 -' 1 \leq i_2$.

(8) $i -' 2 = i -' 1 -' 1$.

(9) If $i_1 + 1 \leq i_2$, then $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leq i_2$.

(10) Suppose $i_1 + 2 \leq i_2$ or $i_1 + 1 + 1 \leq i_2$. Then $i_1 + 1 < i_2$ and $(i_1 + 1) -' 1 < i_2$ and $(i_1 + 1) -' 2 < i_2$ and $i_1 + 1 \leq i_2$ and $(i_1 -' 1) + 1 < i_2$ and $((i_1 -' 1) + 1) -' 1 < i_2$ and $i_1 < i_2$ and $i_1 -' 1 < i_2$ and $i_1 -' 2 < i_2$ and $i_1 \leq i_2$.

[1]The work has been done while the second author was visiting Nagano in autumn 1996.

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(11) If $i_1 \leq i_2$ or $i_1 \leq i_2 -' 1$, then $i_1 < i_2 + 1$ and $i_1 \leq i_2 + 1$ and $i_1 < i_2 + 1 + 1$ and $i_1 \leq i_2 + 1 + 1$ and $i_1 < i_2 + 2$ and $i_1 \leq i_2 + 2$.

(12) If $i_1 < i_2$ or $i_1 + 1 \leq i_2$, then $i_1 \leq i_2 -' 1$.

(13) If $i \geq i_1$, then $i \geq i_1 -' i_2$.

(14) If $1 \leq i$ and $1 \leq i_1 -' i$, then $i_1 -' i < i_1$.

We follow the rules: n, i, i_1, j denote natural numbers and D denotes a non empty set.
Next we state several propositions:

(15) For all finite sequences p, q such that $\operatorname{len} p < i$ but $i \leq \operatorname{len} p + \operatorname{len} q$ or $i \leq \operatorname{len}(p \frown q)$ holds $(p \frown q)(i) = q(i - \operatorname{len} p)$.

(16) For every set x and for every finite sequence f holds $(f \frown \langle x \rangle)(\operatorname{len} f + 1) = x$ and $(\langle x \rangle \frown f)(1) = x$.

(17) Let x be a set and f be a finite sequence of elements of D. Suppose $1 \leq \operatorname{len} f$. Then $(f \frown \langle x \rangle)(1) = f(1)$ and $(f \frown \langle x \rangle)(1) = f_1$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = f(\operatorname{len} f)$ and $(\langle x \rangle \frown f)(\operatorname{len} f + 1) = f_{\operatorname{len} f}$.

(18) For every finite sequence f such that $\operatorname{len} f = 1$ holds $\operatorname{Rev}(f) = f$.

(19) For every finite sequence f of elements of D and for every natural number k holds $\operatorname{len}(f_{\downharpoonleft k}) = \operatorname{len} f -' k$.

(20) Let D be a set, f be a finite sequence of elements of D, and k be a natural number. If $k \leq n$, then $(f \restriction n)(k) = f(k)$.

(21) For every finite sequence f of elements of D and for all natural numbers l_1, l_2 holds $f_{\downharpoonleft l_1} \restriction (l_2 -' l_1) = (f \restriction l_2)_{\downharpoonleft l_1}$.

2. Middle Function for Finite Sequences

Let us consider D, let f be a finite sequence of elements of D, and let k_1, k_2 be natural numbers. The functor $\operatorname{mid}(f, k_1, k_2)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 1) $$\operatorname{mid}(f, k_1, k_2) = \begin{cases} f_{\downharpoonleft k_1 -' 1} \restriction ((k_2 -' k_1) + 1), & \text{if } k_1 \leq k_2, \\ \operatorname{Rev}(f_{\downharpoonleft k_2 -' 1} \restriction ((k_1 -' k_2) + 1)), & \text{otherwise.} \end{cases}$$

The following propositions are true:

(22) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. If $1 \leq k_1$ and $k_1 \leq \operatorname{len} f$ and $1 \leq k_2$ and $k_2 \leq \operatorname{len} f$, then $\operatorname{Rev}(\operatorname{mid}(f, k_1, k_2)) = \operatorname{mid}(\operatorname{Rev}(f), (\operatorname{len} f -' k_2) + 1, (\operatorname{len} f -' k_1) + 1)$.

(23) Let n, m be natural numbers and f be a finite sequence of elements of D. If $1 \leq m$ and $m + n \leq \operatorname{len} f$, then $f_{\downharpoonleft n}(m) = f(m + n)$.

(24) Let i be a natural number and f be a finite sequence of elements of D. If $1 \leq i$ and $i \leq \operatorname{len} f$, then $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(25) For every finite sequence f of elements of D and for every natural number k such that $1 \leq k$ holds $\operatorname{mid}(f, 1, k) = f \restriction k$.

(26) For every finite sequence f of elements of D and for every natural number k such that $k \leq \operatorname{len} f$ holds $\operatorname{mid}(f, k, \operatorname{len} f) = f_{\downharpoonleft k -' 1}$.

(27) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. Suppose $1 \leq k_1$ and $k_1 \leq \operatorname{len} f$ and $1 \leq k_2$ and $k_2 \leq \operatorname{len} f$. Then

(i) $(\operatorname{mid}(f,k_1,k_2))(1) = f(k_1)$,

(ii) if $k_1 \leq k_2$, then $\operatorname{len}\operatorname{mid}(f,k_1,k_2) = (k_2 -' k_1) + 1$ and for every natural number i such that $1 \leq i$ and $i \leq \operatorname{len}\operatorname{mid}(f,k_1,k_2)$ holds $(\operatorname{mid}(f,k_1,k_2))(i) = f((i+k_1) -' 1)$, and

(iii) if $k_1 > k_2$, then $\operatorname{len}\operatorname{mid}(f,k_1,k_2) = (k_1 -' k_2) + 1$ and for every natural number i such that $1 \leq i$ and $i \leq \operatorname{len}\operatorname{mid}(f,k_1,k_2)$ holds $(\operatorname{mid}(f,k_1,k_2))(i) = f((k_1 -' i) + 1)$.

(28) For every finite sequence f of elements of D and for all natural numbers k_1, k_2 holds $\operatorname{rng}\operatorname{mid}(f,k_1,k_2) \subseteq \operatorname{rng} f$.

(29) For every finite sequence f of elements of D such that $1 \leq \operatorname{len} f$ holds $\operatorname{mid}(f,1,\operatorname{len} f) = f$.

(30) For every finite sequence f of elements of D such that $1 \leq \operatorname{len} f$ holds $\operatorname{mid}(f,\operatorname{len} f,1) = \operatorname{Rev}(f)$.

(31) Let f be a finite sequence of elements of D and k_1, k_2, i be natural numbers. Suppose $1 \leq k_1$ and $k_1 \leq k_2$ and $k_2 \leq \operatorname{len} f$ and $1 \leq i$ and $i \leq (k_2 -' k_1) + 1$ or $i \leq (k_2 - k_1) + 1$ or $i \leq (k_2 + 1) - k_1$. Then $(\operatorname{mid}(f,k_1,k_2))(i) = f((i+k_1) -' 1)$ and $(\operatorname{mid}(f,k_1,k_2))(i) = f((i -' 1) + k_1)$ and $(\operatorname{mid}(f,k_1,k_2))(i) = f((i+k_1) - 1)$ and $(\operatorname{mid}(f,k_1,k_2))(i) = f((i-1) + k_1)$.

(32) Let f be a finite sequence of elements of D and k, i be natural numbers. If $1 \leq i$ and $i \leq k$ and $k \leq \operatorname{len} f$, then $(\operatorname{mid}(f,1,k))(i) = f(i)$.

(33) Let f be a finite sequence of elements of D and k_1, k_2 be natural numbers. If $1 \leq k_1$ and $k_1 \leq k_2$ and $k_2 \leq \operatorname{len} f$, then $\operatorname{len}\operatorname{mid}(f,k_1,k_2) \leq \operatorname{len} f$.

(34) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $2 \leq \operatorname{len} f$ holds $f(1) \in \widetilde{\mathcal{L}}(f)$ and $f_1 \in \widetilde{\mathcal{L}}(f)$ and $f(\operatorname{len} f) \in \widetilde{\mathcal{L}}(f)$ and $f_{\operatorname{len} f} \in \widetilde{\mathcal{L}}(f)$.

(35) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $q_1 \in \mathcal{L}(p_1,p_2)$ but $q_2 \in \mathcal{L}(p_1,p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(36) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ but $\mathcal{L}(q_1,q_2) \subseteq \mathcal{L}(p_1,p_2)$ holds $(q_1)_{\mathbf{1}} = (q_2)_{\mathbf{1}}$ or $(q_1)_{\mathbf{2}} = (q_2)_{\mathbf{2}}$.

(37) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. If $2 \leq n$ and f is a special sequence, then $f \restriction n$ is a special sequence.

(38) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n \leq \operatorname{len} f$ and $2 \leq \operatorname{len} f -' n$ and f is a special sequence. Then $f_{\lfloor n}$ is a special sequence.

(39) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and k_1, k_2 be natural numbers. Suppose f is a special sequence and $1 \leq k_1$ and $k_1 \leq \operatorname{len} f$ and $1 \leq k_2$ and $k_2 \leq \operatorname{len} f$ and $k_1 \neq k_2$. Then $\operatorname{mid}(f,k_1,k_2)$ is a special sequence.

3. A CONCEPT OF INDEX FOR FINITE SEQUENCES IN $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that $p \in \widetilde{\mathcal{L}}(f)$. The functor $\operatorname{Index}(p,f)$ yielding a natural number is defined by:

(Def. 2) There exists a non empty subset S of $\mathbb{N}$ such that $\operatorname{Index}(p,f) = \min S$ and $S = \{i : p \in \mathcal{L}(f,i)\}$.

One can prove the following propositions:

(40) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. If $p \in \mathcal{L}(f,i)$, then $\operatorname{Index}(p,f) \leq i$.

(41) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $1 \leq \mathrm{Index}(p, f)$ and $\mathrm{Index}(p, f) < \mathrm{len}\, f$.

(42) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $p \in \mathcal{L}(f, \mathrm{Index}(p, f))$.

(43) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \mathcal{L}(f, 1)$ holds $\mathrm{Index}(p, f) = 1$.

(44) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $\mathrm{len}\, f \geq 2$ holds $\mathrm{Index}(f_1, f) = 1$.

(45) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and given i_1. If f is a special sequence and $1 < i_1$ and $i_1 \leq \mathrm{len}\, f$ and $p = f(i_1)$, then $\mathrm{Index}(p, f) + 1 = i_1$.

(46) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and given i_1. If f is a special sequence and $p \in \mathcal{L}(f, i_1)$, then $i_1 = \mathrm{Index}(p, f)$ or $i_1 = \mathrm{Index}(p, f) + 1$.

(47) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and given i_1. If f is a special sequence and $i_1 + 1 \leq \mathrm{len}\, f$ and $p \in \mathcal{L}(f, i_1)$ and $p \neq f(i_1)$, then $i_1 = \mathrm{Index}(p, f)$.

Let g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. We say that g is a special sequence joining p_1, p_2 if and only if:

(Def. 3) g is a special sequence and $g(1) = p_1$ and $g(\mathrm{len}\, g) = p_2$.

One can prove the following propositions:

(48) Let g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose g is a special sequence joining p_1, p_2. Then $\mathrm{Rev}(g)$ is a special sequence joining p_2, p_1.

(49) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and given j. If $p \in \widetilde{\mathcal{L}}(f)$ and $g = \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f)$ and $1 \leq j$ and $j + 1 \leq \mathrm{len}\, g$, then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, (\mathrm{Index}(p, f) + j) -' 1)$.

(50) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{Index}(p, f) + 1)$ and $g = \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f)$. Then g is a special sequence joining p, $f_{\mathrm{len}\, f}$.

(51) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and given j. If $p \in \widetilde{\mathcal{L}}(f)$ and $1 \leq j$ and $j + 1 \leq \mathrm{len}\, g$ and $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$, then $\mathcal{L}(g, j) \subseteq \mathcal{L}(f, j)$.

(52) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$ and $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$. Then g is a special sequence joining f_1, p.

4. Left and Right Cutting Functions for Finite Sequences in $\mathcal{E}^2_{\rm T}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and let p be a point of $\mathcal{E}^2_{\rm T}$. The functor $\downharpoonleft p, f$ yielding a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 4) $$\downharpoonleft p, f = \begin{cases} \langle p \rangle \frown \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f), & \text{if } p \neq f(\mathrm{Index}(p, f) + 1), \\ \mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f), & \text{otherwise.} \end{cases}$$

The functor $\downharpoonright f, p$ yields a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 5) $$\downharpoonright f, p = \begin{cases} (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle, & \text{if } p \neq f(1), \\ \langle p \rangle, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(53) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p = f(\mathrm{Index}(p, f) + 1)$ and $p \neq f(\mathrm{len}\, f)$. Then $\mathrm{Index}(p, \mathrm{Rev}(f)) + \mathrm{Index}(p, f) + 1 = \mathrm{len}\, f$.

(54) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{Index}(p, f) + 1)$, then $\mathrm{Index}(p, \mathrm{Rev}(f)) + \mathrm{Index}(p, f) = \mathrm{len}\, f$.

(55) Let given D, f be a finite sequence of elements of D, k be a natural number, and p be an element of D. Then $(\langle p \rangle ^\frown f){\restriction}(k+1) = \langle p \rangle ^\frown (f{\restriction}k)$.

(56) Let given D, f be a non empty finite sequence of elements of D, and k_1, k_2 be natural numbers. If $k_1 < k_2$ and $k_1 \in \mathrm{dom}\, f$, then $\mathrm{mid}(f, k_1, k_2) = \langle f(k_1) \rangle ^\frown \mathrm{mid}(f, k_1 + 1, k_2)$.

Let f be a non empty finite sequence. One can check that $\mathrm{Rev}(f)$ is non empty.
One can prove the following propositions:

(57) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonright p, \mathrm{Rev}(f) = \mathrm{Rev}(\downharpoonleft f, p)$.

(58) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $p \in \widetilde{\mathcal{L}}(f)$. Then

(i) $(\downharpoonright p, f)(1) = p$, and

(ii) for every i such that $1 < i$ and $i \leq \mathrm{len} \downharpoonright p, f$ holds if $p = f(\mathrm{Index}(p, f) + 1)$, then $(\downharpoonright p, f)(i) = f(\mathrm{Index}(p, f) + i)$ and if $p \neq f(\mathrm{Index}(p, f) + 1)$, then $(\downharpoonright p, f)(i) = f((\mathrm{Index}(p, f) + i) - 1)$.

(59) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then $(\downharpoonleft f, p)(\mathrm{len} \downharpoonleft f, p) = p$ and for every i such that $1 \leq i$ and $i \leq \mathrm{Index}(p, f)$ holds $(\downharpoonleft f, p)(i) = f(i)$.

(60) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$ such that f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then

(i) if $p \neq f(1)$, then $\mathrm{len} \downharpoonleft f, p = \mathrm{Index}(p, f) + 1$, and

(ii) if $p = f(1)$, then $\mathrm{len} \downharpoonleft f, p = \mathrm{Index}(p, f)$.

(61) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$ such that f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then

(i) if $p = f(\mathrm{Index}(p, f) + 1)$, then $\mathrm{len} \downharpoonright p, f = \mathrm{len}\, f - \mathrm{Index}(p, f)$, and

(ii) if $p \neq f(\mathrm{Index}(p, f) + 1)$, then $\mathrm{len} \downharpoonright p, f = (\mathrm{len}\, f - \mathrm{Index}(p, f)) + 1$.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. The predicate $q_1 \leq_{p_1, p_2} q_2$ is defined by the conditions (Def. 6).

(Def. 6)(i) $q_1 \in \mathcal{L}(p_1, p_2)$,

(ii) $q_2 \in \mathcal{L}(p_1, p_2)$, and

(iii) for all real numbers r_1, r_2 such that $0 \leq r_1$ and $r_1 \leq 1$ and $q_1 = (1 - r_1) \cdot p_1 + r_1 \cdot p_2$ and $0 \leq r_2$ and $r_2 \leq 1$ and $q_2 = (1 - r_2) \cdot p_1 + r_2 \cdot p_2$ holds $r_1 \leq r_2$.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. The predicate $q_1 <_{p_1, p_2} q_2$ is defined as follows:

(Def. 7) $q_1 \leq_{p_1, p_2} q_2$ and $q_1 \neq q_2$.

Next we state several propositions:

(62) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\rm T}$ such that $q_1 \leq_{p_1, p_2} q_2$ and $q_2 \leq_{p_1, p_2} q_1$ holds $q_1 = q_2$.

(63) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$ and $p_1 \neq p_2$ holds $q_1 \leq_{p_1,p_2} q_2$ or $q_2 <_{p_1,p_2} q_1$ but $q_1 \leq_{p_1,p_2} q_2$ but $q_2 <_{p_1,p_2} q_1$.

(64) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{Index}(p, f) < \mathrm{Index}(q, f)$, then $q \in \widetilde{\mathcal{L}}(\downharpoonright p, f)$.

(65) For all points p, q, p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \leq_{p_1,p_2} q$ holds $q \in \mathcal{L}(p, p_2)$ and $p \in \mathcal{L}(p_1, q)$.

(66) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $\mathrm{Index}(p, f) = \mathrm{Index}(q, f)$ and $p \leq_{f_{\mathrm{Index}(p,f)}, f_{\mathrm{Index}(p,f)+1}} q$. Then $q \in \widetilde{\mathcal{L}}(\downharpoonright p, f)$.

5. Cutting Both Sides of a Finite Sequence and a Discussion of Speciality of Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\downharpoonright\downharpoonleft p, f, q$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 8) $\downharpoonright\downharpoonleft p, f, q = \begin{cases} \downharpoonleft\downharpoonright p, f, q, \text{ if } p \in \widetilde{\mathcal{L}}(f) \text{ and } q \in \widetilde{\mathcal{L}}(f) \text{ and } \mathrm{Index}(p, f) < \mathrm{Index}(q, f) \text{ or } \mathrm{Index}(p, f) = \mathrm{Index}(q, f) \text{ and} \\ \mathrm{Rev}(\downharpoonleft\downharpoonright q, f, p), \text{ otherwise.} \end{cases}$

One can prove the following propositions:

(67) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonleft f, p$ is a special sequence joining f_1, p.

(68) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\downharpoonright p, f$ is a special sequence joining p, $f_{\mathrm{len}\, f}$.

(69) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\downharpoonright p, f$ is a special sequence.

(70) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonleft f, p$ is a special sequence.

(71) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\downharpoonright\downharpoonleft p, f, q$ is a special sequence joining p, q.

(72) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$. Then $\downharpoonright\downharpoonleft p, f, q$ is a special sequence.

(73) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence.

(74) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f(\mathrm{len}\, f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $f \frown \mathrm{mid}(g, 2, \mathrm{len}\, g)$ is a special sequence joining f_1, $g_{\mathrm{len}\, g}$.

(75) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(f_{\restriction n}) \subseteq \widetilde{\mathcal{L}}(f)$.

(76) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonleft f, p) \subseteq \widetilde{\mathcal{L}}(f)$.

(77) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence, then $\widetilde{\mathcal{L}}(\downharpoonleft p, f) \subseteq \widetilde{\mathcal{L}}(f)$.

(78) Let f, g be non empty finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\operatorname{len} f)$. Then $(\downharpoonleft p, f) \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence joining p, $g_{\operatorname{len} g}$.

(79) Let f, g be non empty finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq f(\operatorname{len} f)$. Then $(\downharpoonleft p, f) \frown \operatorname{mid}(g, 2, \operatorname{len} g)$ is a special sequence.

(80) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown g$ is a special sequence.

(81) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown g$ is a special sequence joining f_1, $g_{\operatorname{len} g}$.

(82) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonright g, p$ is a special sequence joining f_1, p.

(83) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $f(\operatorname{len} f) = g(1)$ and $p \in \widetilde{\mathcal{L}}(g)$ and f is a special sequence and g is a special sequence and $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g) = \{g(1)\}$ and $p \neq g(1)$. Then $(\operatorname{mid}(f, 1, \operatorname{len} f -' 1)) \frown \downharpoonright g, p$ is a special sequence.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal2.html.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Some properties of restrictions of finite sequences. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/finseq_5.html.

[7] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[8] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm T}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[9] Agata Darmochwał and Andrzej Trybulec. Similarity of formulae. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/cqc_sim1.html.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[11] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfinseq.html.

[12] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/binarith.html.

[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[15] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[16] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[17] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[18] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

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