JOURNAL OF FORMALIZED MATHEMATICS
Volume 13, Released 2001, Published 2003
Inst. of Computer Science, Univ. of Białystok

Upper and Lower Sequence on the Cage. Part II[1]

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MML Identifier: JORDAN1G.

WWW: http://mizar.org/JFM/Vol13/jordan1g.html

The articles [24], [28], [13], [4], [2], [27], [5], [14], [3], [25], [23], [1], [22], [9], [10], [7], [26], [16], [11], [20], [18], [19], [6], [12], [21], [8], [15], and [17] provide the notation and terminology for this paper.

In this paper n is a natural number.

Let us note that there exists a finite sequence which is trivial.

Next we state the proposition

(1) For every trivial finite sequence f holds f is empty or there exists a set x such that $f = \langle x \rangle$.

Let p be a non trivial finite sequence. Observe that $\mathrm{Rev}(p)$ is non trivial.

One can prove the following propositions:

(2) Let D be a non empty set, f be a finite sequence of elements of D, G be a matrix over D, and p be a set. Suppose f is a sequence which elements belong to G. Then $f -: p$ is a sequence which elements belong to G.

(3) Let D be a non empty set, f be a finite sequence of elements of D, G be a matrix over D, and p be an element of D. Suppose $p \in \mathrm{rng}\, f$. Suppose f is a sequence which elements belong to G. Then $f :- p$ is a sequence which elements belong to G.

(4) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{UpperSeq}(C,n)$ is a sequence which elements belong to $\mathrm{Gauge}(C,n)$.

(5) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{LowerSeq}(C,n)$ is a sequence which elements belong to $\mathrm{Gauge}(C,n)$.

Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. One can verify that $\mathrm{UpperSeq}(C,n)$ is standard and $\mathrm{LowerSeq}(C,n)$ is standard.

One can prove the following propositions:

(6) Let G be a column **Y**-constant line **Y**-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$ and i_1, i_2, j_1, j_2 be natural numbers. Suppose $\langle i_1, j_1 \rangle \in$ the indices of G and $\langle i_2, j_2 \rangle \in$ the indices of G. If $(G \circ (i_1, j_1))_{\mathbf{2}} = (G \circ (i_2, j_2))_{\mathbf{2}}$, then $j_1 = j_2$.

(7) Let G be a line **X**-constant column **X**-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$ and i_1, i_2, j_1, j_2 be natural numbers. Suppose $\langle i_1, j_1 \rangle \in$ the indices of G and $\langle i_2, j_2 \rangle \in$ the indices of G. If $(G \circ (i_1, j_1))_{\mathbf{1}} = (G \circ (i_2, j_2))_{\mathbf{1}}$, then $i_1 = i_2$.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

(16)[1] Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm N}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm N}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm N}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm N}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then $({\rm N}_{\rm min}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}} < ({\rm N}_{\rm max}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}}$.

(17) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm N}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm N}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm N}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm N}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then ${\rm N}_{\rm min}(\widetilde{\mathcal{L}}(f)) \neq {\rm N}_{\rm max}(\widetilde{\mathcal{L}}(f))$.

(18) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm S}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm S}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm S}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm S}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then $({\rm S}_{\rm min}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}} < ({\rm S}_{\rm max}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}}$.

(19) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm S}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm S}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm S}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm S}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then ${\rm S}_{\rm min}(\widetilde{\mathcal{L}}(f)) \neq {\rm S}_{\rm max}(\widetilde{\mathcal{L}}(f))$.

(20) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm W}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm W}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm W}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm W}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then $({\rm W}_{\rm min}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}} < ({\rm W}_{\rm max}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}}$.

(21) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm W}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm W}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm W}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm W}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then ${\rm W}_{\rm min}(\widetilde{\mathcal{L}}(f)) \neq {\rm W}_{\rm max}(\widetilde{\mathcal{L}}(f))$.

(22) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm E}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm E}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm E}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm E}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then $({\rm E}_{\rm min}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}} < ({\rm E}_{\rm max}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}}$.

(23) Let f be a standard special unfolded non trivial finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If $f_1 \neq {\rm E}_{\rm min}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm E}_{\rm min}(\widetilde{\mathcal{L}}(f))$ or $f_1 \neq {\rm E}_{\rm max}(\widetilde{\mathcal{L}}(f))$ and $f_{{\rm len}\, f} \neq {\rm E}_{\rm max}(\widetilde{\mathcal{L}}(f))$, then ${\rm E}_{\rm min}(\widetilde{\mathcal{L}}(f)) \neq {\rm E}_{\rm max}(\widetilde{\mathcal{L}}(f))$.

(24) Let D be a non empty set, f be a finite sequence of elements of D, and p, q be elements of D. If $p \in {\rm rng}\, f$ and $q \in {\rm rng}\, f$ and $q \hookleftarrow f \leq p \hookleftarrow f$, then $(f -: p) :- q = (f :- q) -: p$.

(25) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Then $\widetilde{\mathcal{L}}({\rm Cage}(C,n) -: {\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \cap \widetilde{\mathcal{L}}({\rm Cage}(C,n) :- {\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) = \{{\rm N}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n))), {\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))\}$.

(26) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds ${\rm LowerSeq}(C,n) = ({\rm Cage}(C,n) \circlearrowleft {\rm E}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) -: {\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))$.

(27) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n) = 1$.

(28) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n) < ({\rm W}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n)$.

(29) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm W}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n) \leq ({\rm N}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n)$.

(30) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm N}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n) < ({\rm N}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n)$.

(31) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm N}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n) \leq ({\rm E}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \hookleftarrow {\rm UpperSeq}(C,n)$.

[1] The propositions (8)–(15) have been removed.

(32) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm E}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp$ ${\rm UpperSeq}(C,n) = {\rm len}\,{\rm UpperSeq}(C,n)$.

(33) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm E}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp$ ${\rm LowerSeq}(C,n) = 1$.

(34) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm E}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n) < ({\rm E}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n)$.

(35) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm E}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n) \leq ({\rm S}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n)$.

(36) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm S}_{\rm max}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n) < ({\rm S}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n)$.

(37) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm S}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n) \leq ({\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n)$.

(38) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $({\rm W}_{\rm min}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))) \leftarrow\!\!\!\!\wp {\rm LowerSeq}(C,n) = {\rm len}\,{\rm LowerSeq}(C,n)$.

(39) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(({\rm UpperSeq}(C,n))_2)_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))$.

(40) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $(({\rm LowerSeq}(C,n))_2)_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}({\rm Cage}(C,n)))$.

(41) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\text{W-bound}(\widetilde{\mathcal{L}}({\rm Cage}(C,n))) + \text{E-bound}(\widetilde{\mathcal{L}}({\rm Cage}(C,n))) = \text{W-bound}(C) + \text{E-bound}(C)$.

(42) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\text{S-bound}(\widetilde{\mathcal{L}}({\rm Cage}(C,n))) + \text{N-bound}(\widetilde{\mathcal{L}}({\rm Cage}(C,n))) = \text{S-bound}(C) + \text{N-bound}(C)$.

(43) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n, i be natural numbers. If $1 \leq i$ and $i \leq {\rm width}\,{\rm Gauge}(C,n)$ and $n > 0$, then $({\rm Gauge}(C,n) \circ ({\rm Center}\,{\rm Gauge}(C,n), i))_{\mathbf{1}} = \frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}$.

(44) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n, i be natural numbers. If $1 \leq i$ and $i \leq {\rm len}\,{\rm Gauge}(C,n)$ and $n > 0$, then $({\rm Gauge}(C,n) \circ (i, {\rm Center}\,{\rm Gauge}(C,n)))_{\mathbf{2}} = \frac{\text{S-bound}(C)+\text{N-bound}(C)}{2}$.

(45) Let f be a S-sequence in $\mathbb{R}^2$ and k_1, k_2 be natural numbers. If $1 \leq k_1$ and $k_1 \leq {\rm len}\,f$ and $1 \leq k_2$ and $k_2 \leq {\rm len}\,f$ and $f_1 \in \widetilde{\mathcal{L}}({\rm mid}(f, k_1, k_2))$, then $k_1 = 1$ or $k_2 = 1$.

(46) Let f be a S-sequence in $\mathbb{R}^2$ and k_1, k_2 be natural numbers. If $1 \leq k_1$ and $k_1 \leq {\rm len}\,f$ and $1 \leq k_2$ and $k_2 \leq {\rm len}\,f$ and $f_{{\rm len}\,f} \in \widetilde{\mathcal{L}}({\rm mid}(f, k_1, k_2))$, then $k_1 = {\rm len}\,f$ or $k_2 = {\rm len}\,f$.

(47) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Then ${\rm rng}\,{\rm UpperSeq}(C,n) \subseteq {\rm rng}\,{\rm Cage}(C,n)$ and ${\rm rng}\,{\rm LowerSeq}(C,n) \subseteq {\rm rng}\,{\rm Cage}(C,n)$.

(48) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds ${\rm UpperSeq}(C,n)$ is a h.c. for ${\rm Cage}(C,n)$.

(49) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds ${\rm Rev}({\rm LowerSeq}(C,n))$ is a h.c. for ${\rm Cage}(C,n)$.

(50) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 < i$ and $i \leq {\rm len}\,{\rm Gauge}(C,n)$, then ${\rm Gauge}(C,n) \circ (i,1) \notin {\rm rng}\,{\rm UpperSeq}(C,n)$.

(51) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 \leq i$ and $i < \operatorname{len}\operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i, \operatorname{width}\operatorname{Gauge}(C,n)) \notin \operatorname{rng}\operatorname{LowerSeq}(C,n)$.

(52) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 < i$ and $i \leq \operatorname{len}\operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i,1) \notin \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n))$.

(53) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i be a natural number. If $1 \leq i$ and $i < \operatorname{len}\operatorname{Gauge}(C,n)$, then $\operatorname{Gauge}(C,n) \circ (i, \operatorname{width}\operatorname{Gauge}(C,n)) \notin \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$.

(54) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i, j be natural numbers. Suppose $1 \leq i$ and $i \leq \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leq j$ and $j \leq \operatorname{width}\operatorname{Gauge}(C,n)$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,1), \operatorname{Gauge}(C,n) \circ (i,j))$ meets $\widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$.

(55) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. If $n > 0$, then $\operatorname{FPoint}(\widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\frac{\text{W-bound}(\ldots}{\ldots}$ $\operatorname{rng}\operatorname{UpperSeq}(C,n)$.

(56) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. If $n > 0$, then $\operatorname{LPoint}(\widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\frac{\text{W-bound}(\ldots}{\ldots}$ $\operatorname{rng}\operatorname{LowerSeq}(C,n)$.

(57) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \operatorname{rng} f$ holds $\downharpoonleft f, p = \operatorname{mid}(f, 1, p \leftarrow\!\!\!\rightarrow f)$.

(58) Let f be a S-sequence in $\mathbb{R}^2$ and Q be a closed subset of $\mathcal{E}^2_{\rm T}$. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and $f_1 \notin Q$ and $\operatorname{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\operatorname{len} f}, Q) \in \operatorname{rng} f$. Then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, 1, (\operatorname{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\operatorname{len} f}, Q)) \leftarrow\!\!\!\rightarrow f)) \cap Q = \{\operatorname{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\operatorname{len} f}, Q)\}$.

(59) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$. Let k be a natural number. Suppose $1 \leq k$ and $k < (\operatorname{FPoint}(\widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\frac{\text{W-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))+\text{E-bound}(\ldots}{2}$ $\operatorname{UpperSeq}(C,n)$. Then $((\operatorname{UpperSeq}(C,n))_k)_{\mathbf{1}} < \frac{\text{W-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))+\text{E-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))}{2}$.

(60) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$. Let k be a natural number. Suppose $1 \leq k$ and $k < (\operatorname{FPoint}(\widetilde{\mathcal{L}}(\operatorname{Rev}(\operatorname{LowerSeq}(C,n))), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\frac{\text{W-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))+\text{E-b}\ldots}{2}$ $\operatorname{Rev}(\operatorname{LowerSeq}(C,n))$. Then $((\operatorname{Rev}(\operatorname{LowerSeq}(C,n)))_k)_{\mathbf{1}} < \frac{\text{W-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))+\text{E-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))}{2}$.

(61) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$. Let q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $q \in \operatorname{rng}\operatorname{mid}(\operatorname{UpperSeq}(C,n), 2, (\operatorname{FPoint}(\widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\frac{\text{W-}\ldots}{\ldots}$ $\operatorname{UpperSeq}(C,n))$. Then $q_{\mathbf{1}} \leq \frac{\text{W-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))+\text{E-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))}{2}$.

(62) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$. Then $(\operatorname{FPoint}(\widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\ldots$ $(\operatorname{LPoint}(\widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)), \operatorname{E}_{\max}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{W}_{\min}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))), \operatorname{VerticalLine}(\frac{\text{W-bound}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))+\text{E-bound}(\ldots}{2}$

(63) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. If $n > 0$, then $\widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)) = \operatorname{UpperArc}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))$.

(64) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. If $n > 0$, then $\widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)) = \operatorname{LowerArc}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))$.

(65) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Suppose $n > 0$. Let i, j be natural numbers. Suppose $1 \leq i$ and $i \leq \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leq j$ and $j \leq \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,1), \mathrm{Gauge}(C,n) \circ (i,j))$ meets $\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

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Received September 28, 2001

Published January 2, 2004