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Upper and Lower Sequence of a Cage[1]

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The articles [18], [2], [4], [3], [21], [5], [12], [19], [15], [17], [11], [1], [16], [8], [9], [6], [20], [10], [13], [14], and [7] provide the notation and terminology for this paper.

1. Preliminaries

In this paper n denotes a natural number.

The following propositions are true:

(1) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{N-bound}(X) \leq \text{N-bound}(Y)$.

(2) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{E-bound}(X) \leq \text{E-bound}(Y)$.

(3) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{S-bound}(X) \geq \text{S-bound}(Y)$.

(4) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds $\text{W-bound}(X) \geq \text{W-bound}(Y)$.

(5) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is in the area of g. Let p be an element of $\mathcal{E}^2_{\rm T}$. If $p \in \text{rng} f$, then $f -: p$ is in the area of g.

(6) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is in the area of g. Let p be an element of $\mathcal{E}^2_{\rm T}$. If $p \in \text{rng} f$, then $f :- p$ is in the area of g.

(7) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\downharpoonleft p, f \neq \emptyset$.

(8) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $\text{len} \downharpoonleft f, p \geq 2$, then $f(1) \in \widetilde{\mathcal{L}}(\downharpoonleft f, p)$.

(9) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $f(1) \notin \widetilde{\mathcal{L}}(\text{mid}(f, \text{Index}(p, f)+1, \text{len} f))$.

(10) For all natural numbers i, j, m, n such that $i + j = m + n$ and $i \leq m$ and $j \leq n$ holds $i = m$.

(11) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $f(1) \in \widetilde{\mathcal{L}}(\downharpoonleft p, f)$, then $f(1) = p$.

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2. About Upper and Lower Sequence of a Cage

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and let n be a natural number. The functor $\mathrm{UpperSeq}(C,n)$ yielding a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) $\mathrm{UpperSeq}(C,n) = (\mathrm{Cage}(C,n) \circlearrowleft \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))) -: \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

Next we state the proposition

(12) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) = (\mathrm{E}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))) \hookleftarrow (\mathrm{Cage}(C,n) \circlearrowleft \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and let n be a natural number. The functor $\mathrm{LowerSeq}(C,n)$ yields a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 2) $\mathrm{LowerSeq}(C,n) = (\mathrm{Cage}(C,n) \circlearrowleft \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))) :- \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

We now state the proposition

(13) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Then $\mathrm{len}\,\mathrm{LowerSeq}(C,n) = (\mathrm{len}(\mathrm{Cage}(C,n) \circlearrowleft \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))) - (\mathrm{E}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))) \hookleftarrow (\mathrm{Cage}(C,n) \circlearrowleft \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))) + 1$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and let n be a natural number. Note that $\mathrm{UpperSeq}(C,n)$ is non empty and $\mathrm{LowerSeq}(C,n)$ is non empty.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and let n be a natural number. One can verify that $\mathrm{UpperSeq}(C,n)$ is one-to-one, special, unfolded, and s.n.c. and $\mathrm{LowerSeq}(C,n)$ is one-to-one, special, unfolded, and s.n.c..

The following propositions are true:

(14) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) + \mathrm{len}\,\mathrm{LowerSeq}(C,n) = \mathrm{len}\,\mathrm{Cage}(C,n) + 1$.

(15) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\mathrm{Cage}(C,n) \circlearrowleft \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) = \mathrm{UpperSeq}(C,n) \frown \mathrm{LowerSeq}(C,n)$.

(16) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n) \frown \mathrm{LowerSeq}(C,n))$.

(17) For every compact non vertical non horizontal non empty subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) \cup \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(18) For every simple closed curve P holds $\mathrm{W}_{\min}(P) \neq \mathrm{E}_{\min}(P)$.

(19) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) \geq 3$ and $\mathrm{len}\,\mathrm{LowerSeq}(C,n) \geq 3$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and let n be a natural number. One can verify that $\mathrm{UpperSeq}(C,n)$ is special sequence and $\mathrm{LowerSeq}(C,n)$ is special sequence.

One can prove the following propositions:

(20) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))), \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))\}$.

(21) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{UpperSeq}(C,n)$ is in the area of $\mathrm{Cage}(C,n)$.

(22) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{LowerSeq}(C,n)$ is in the area of $\mathrm{Cage}(C,n)$.

(23) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds $((\mathrm{Cage}(C,n))_2)_{\mathbf{2}} = \text{N-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(24) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and k be a natural number. If $1 \leq k$ and $k+1 \leq \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(25) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and k be a natural number. If $1 \leq k$ and $k+1 \leq \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{S}_{\max}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{2}} = \text{S-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(26) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and k be a natural number. If $1 \leq k$ and $k+1 \leq \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

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