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General Fashoda Meet Theorem for Unit Circle

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Summary. Outside and inside Fashoda theorems are proven for points in general position on unit circle. Four points must be ordered in a sense of ordering for simple closed curve. For preparation of proof, the relation between the order and condition of coordinates of points on unit circle is discussed.

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The articles [1], [18], [11], [9], [17], [20], [8], [4], [5], [10], [2], [7], [12], [19], [16], [6], [3], [15], [14], and [13] provide the notation and terminology for this paper.

1. Preliminaries

In this paper x, a denote real numbers.

One can prove the following propositions:

(1) If $a \geq 0$ and $(x-a)\cdot(x+a) \geq 0$, then $-a \geq x$ or $x \geq a$.

(2) If $a \leq 0$ and $x < a$, then $x^{\mathbf{2}} > a^{\mathbf{2}}$.

(3) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p| \leq 1$ holds $-1 \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq 1$ and $-1 \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq 1$.

(4) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p| \leq 1$ and $p_{\mathbf{1}} \neq 0$ and $p_{\mathbf{2}} \neq 0$ holds $-1 < p_{\mathbf{1}}$ and $p_{\mathbf{1}} < 1$ and $-1 < p_{\mathbf{2}}$ and $p_{\mathbf{2}} < 1$.

(5) Let a, b, d, e, r_3 be real numbers, P_1, P_2 be non empty metric structures, x be an element of P_1, and x_2 be an element of P_2. Suppose $d \leq a$ and $a \leq b$ and $b \leq e$ and $P_1 = [a, b]_{\mathrm{M}}$ and $P_2 = [d, e]_{\mathrm{M}}$ and $x = x_2$ and $x \in$ the carrier of P_1 and $x_2 \in$ the carrier of P_2. Then $\mathrm{Ball}(x, r_3) \subseteq \mathrm{Ball}(x_2, r_3)$.

(6) Let a, b, d, e be real numbers and B be a subset of $[d, e]_{\mathrm{T}}$. If $d \leq a$ and $a \leq b$ and $b \leq e$ and $B = [a, b]$, then $[a, b]_{\mathrm{T}} = [d, e]_{\mathrm{T}} \restriction B$.

(7) For all real numbers a, b and for every subset B of $\mathbb{I}$ such that $0 \leq a$ and $a \leq b$ and $b \leq 1$ and $B = [a, b]$ holds $[a, b]_{\mathrm{T}} = \mathbb{I} \restriction B$.

(8) Let X be a topological structure, Y, Z be non empty topological structures, f be a map from X into Y, and h be a map from Y into Z. If h is a homeomorphism and f is continuous, then $h \cdot f$ is continuous.

(9) Let X, Y, Z be topological structures, f be a map from X into Y, and h be a map from Y into Z. If h is a homeomorphism and f is one-to-one, then $h \cdot f$ is one-to-one.

(10) Let X be a topological structure, S, V be non empty topological structures, B be a non empty subset of S, f be a map from X into $S \restriction B$, g be a map from S into V, and h be a map from X into V. If $h = g \cdot f$ and f is continuous and g is continuous, then h is continuous.

(11) Let a, b, d, e, s_1, s_2, t_1, t_2 be real numbers and h be a map from $[a, b]_{\mathrm{T}}$ into $[d, e]_{\mathrm{T}}$. Suppose h is a homeomorphism and $h(s_1) = t_1$ and $h(s_2) = t_2$ and $h(a) = d$ and $h(b) = e$ and $d \leq e$ and $t_1 \leq t_2$ and $s_1 \in [a,b]$ and $s_2 \in [a,b]$. Then $s_1 \leq s_2$.

(12) Let a, b, d, e, s_1, s_2, t_1, t_2 be real numbers and h be a map from $[a, b]_{\mathrm{T}}$ into $[d, e]_{\mathrm{T}}$. Suppose h is a homeomorphism and $h(s_1) = t_1$ and $h(s_2) = t_2$ and $h(a) = e$ and $h(b) = d$ and $e \geq d$ and $t_1 \geq t_2$ and $s_1 \in [a,b]$ and $s_2 \in [a,b]$. Then $s_1 \leq s_2$.

(13) For every natural number n holds $-0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

2. Fashoda Meet Theorems for Circle in Special Case

One can prove the following propositions:

(14) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $a \neq b$ and $c \neq d$ and $f(O)_{\mathbf{1}} = a$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $f(I)_{\mathbf{1}} = b$ and $c \leq f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq d$ and $g(O)_{\mathbf{2}} = c$ and $a \leq g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leq b$ and $g(I)_{\mathbf{2}} = d$ and $a \leq g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leq b$ and for every point r of $\mathbb{I}$ holds $a \geq f(r)_{\mathbf{1}}$ or $f(r)_{\mathbf{1}} \geq b$ or $c \geq f(r)_{\mathbf{2}}$ or $f(r)_{\mathbf{2}} \geq d$ but $a \geq g(r)_{\mathbf{1}}$ or $g(r)_{\mathbf{1}} \geq b$ or $c \geq g(r)_{\mathbf{2}}$ or $g(r)_{\mathbf{2}} \geq d$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(15) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is continuous and one-to-one. Then there exists a map f_2 from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that $f_2(0) = f(1)$ and $f_2(1) = f(0)$ and $\operatorname{rng} f_2 = \operatorname{rng} f$ and f_2 is continuous and one-to-one.

In the sequel p, q are points of $\mathcal{E}^2_{\mathrm{T}}$.
The following propositions are true:

(16) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \land (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \land (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \land (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \land (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \land (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \land (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \land (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \land (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(17) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \land (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \land (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \land (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \land (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \land (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \land (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \land (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \land (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_4$ and $g(I) \in K_3$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(18) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \land (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \land (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \land (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \land (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \land (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \land (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| =$

$1 \wedge (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(19) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$ and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $C_0 = \{q : |q| \geq 1\}$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(0) = [-1,0]$ and $f(1) = [1,0]$ and $g(1) = [0,1]$ and $g(0) = [0,-1]$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(20) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $C_0 = \{p : |p| \geq 1\}$,

(ii) $|p_1| = 1$,

(iii) $|p_2| = 1$,

(iv) $|p_3| = 1$,

(v) $|p_4| = 1$, and

(vi) there exists a map h from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that h is a homeomorphism and $h^{\circ}C_0 \subseteq C_0$ and $h(p_1) = [-1,0]$ and $h(p_2) = [0,1]$ and $h(p_3) = [1,0]$ and $h(p_4) = [0,-1]$.

Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

3. Properties of Fan Morphisms

One can prove the following propositions:

(21) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} > 0$.

(22) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \geq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} \geq 0$.

(23) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \geq 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$ and $|q| \neq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} \geq 0$ and $p_{\mathbf{1}} < 0$.

(24) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} \geq 0$ and $(q_2)_{\mathbf{2}} \geq 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN(q_1) and $p_2 = c_1$-FanMorphN(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(25) Let s_3 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $q_{\mathbf{1}} > 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_3$-FanMorphE(q), then $p_{\mathbf{1}} > 0$.

(26) Let s_3 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $q_{\mathbf{1}} \geq 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_3$ and $|q| \neq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_3$-FanMorphE(q), then $p_{\mathbf{1}} \geq 0$ and $p_{\mathbf{2}} < 0$.

(27) Let s_3 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $(q_1)_{\mathbf{1}} \geq 0$ and $(q_2)_{\mathbf{1}} \geq 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_3$-FanMorphE(q_1) and $p_2 = s_3$-FanMorphE(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(28) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} < 0$.

(29) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} > c_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} > 0$.

(30) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\bf 2} \leq 0$ and $(q_2)_{\bf 2} \leq 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\bf 1}}{|q_1|} < \frac{(q_2)_{\bf 1}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = c_1$-FanMorphS(q_1) and $p_2 = c_1$-FanMorphS(q_2), then $\frac{(p_1)_{\bf 1}}{|p_1|} < \frac{(p_2)_{\bf 1}}{|p_2|}$.

4. ORDER OF POINTS ON CIRCLE

We now state a number of propositions:

(31) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds W-bound$(P) = -1$ and E-bound$(P) = 1$ and S-bound$(P) = -1$ and N-bound$(P) = 1$.

(32) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds $\mathrm{W_{min}}(P) = [-1, 0]$.

(33) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds $\mathrm{E_{max}}(P) = [1, 0]$.

(34) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\bf 1}$ such that for every point p of $\mathcal{E}^2_{\rm T}$ holds $f(p) = \mathrm{proj1}(p)$ holds f is continuous.

(35) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\bf 1}$ such that for every point p of $\mathcal{E}^2_{\rm T}$ holds $f(p) = \mathrm{proj2}(p)$ holds f is continuous.

(36) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$ holds UpperArc$(P) \subseteq P$ and LowerArc$(P) \subseteq P$.

(37) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$. Then UpperArc$(P) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p \in P \ \wedge\ p_{\bf 2} \geq 0\}$.

(38) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$. Then LowerArc$(P) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p \in P \ \wedge\ p_{\bf 2} \leq 0\}$.

(39) Let a, b, d, e be real numbers. Suppose $a \leq b$ and $e > 0$. Then there exists a map f from $[a, b]_{\rm T}$ into $[e \cdot a + d, e \cdot b + d]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [a, b]$ holds $f(r) = e \cdot r + d$.

(40) Let a, b, d, e be real numbers. Suppose $a \leq b$ and $e < 0$. Then there exists a map f from $[a, b]_{\rm T}$ into $[e \cdot b + d, e \cdot a + d]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [a, b]$ holds $f(r) = e \cdot r + d$.

(41) There exists a map f from $\mathbb{I}$ into $[-1, 1]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [0, 1]$ holds $f(r) = (-2) \cdot r + 1$ and $f(0) = 1$ and $f(1) = -1$.

(42) There exists a map f from $\mathbb{I}$ into $[-1, 1]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [0, 1]$ holds $f(r) = 2 \cdot r - 1$ and $f(0) = -1$ and $f(1) = 1$.

(43) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $[-1, 1]_{\rm T}$ into $(\mathcal{E}^2_{\rm T}) \restriction$LowerArc$(P)$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ LowerArc(P) holds $f(q_{\bf 1}) = q$ and $f(-1) = \mathrm{W_{min}}(P)$ and $f(1) = \mathrm{E_{max}}(P)$.

(44) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $[-1, 1]_{\rm T}$ into $(\mathcal{E}^2_{\rm T}) \restriction$UpperArc$(P)$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ UpperArc(P) holds $f(q_{\bf 1}) = q$ and $f(-1) = \mathrm{W_{min}}(P)$ and $f(1) = \mathrm{E_{max}}(P)$.

(45) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T}) \restriction$LowerArc$(P)$ such that

(i) f is a homeomorphism,

(ii) for all points q_1, q_2 of $\mathcal{E}^2_{\rm T}$ and for all real numbers r_1, r_2 such that $f(r_1) = q_1$ and $f(r_2) = q_2$ and $r_1 \in [0,1]$ and $r_2 \in [0,1]$ holds $r_1 < r_2$ iff $(q_1)_{\mathbf{1}} > (q_2)_{\mathbf{1}}$,

(iii) $f(0) = {\rm E_{max}}(P)$, and

(iv) $f(1) = {\rm W_{min}}(P)$.

(46) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction{\rm UpperArc}(P)$ such that

(i) f is a homeomorphism,

(ii) for all points q_1, q_2 of $\mathcal{E}^2_{\rm T}$ and for all real numbers r_1, r_2 such that $f(r_1) = q_1$ and $f(r_2) = q_2$ and $r_1 \in [0,1]$ and $r_2 \in [0,1]$ holds $r_1 < r_2$ iff $(q_1)_{\mathbf{1}} < (q_2)_{\mathbf{1}}$,

(iii) $f(0) = {\rm W_{min}}(P)$, and

(iv) $f(1) = {\rm E_{max}}(P)$.

(47) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_2 \in {\rm UpperArc}(P)$ and $p_1 \leq_P p_2$, then $p_1 \in {\rm UpperArc}(P)$.

(48) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$. Then $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(49) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$. Then $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(50) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$. Then $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$.

(51) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{2}} \leq 0$ and $(p_2)_{\mathbf{2}} \leq 0$ and $p_1 \neq {\rm W_{min}}(P)$. Then $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$.

(52) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ but $(p_2)_{\mathbf{2}} \geq 0$ or $(p_2)_{\mathbf{1}} \geq 0$ but $p_1 \leq_P p_2$. Then $(p_1)_{\mathbf{2}} \geq 0$ or $(p_1)_{\mathbf{1}} \geq 0$.

(53) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} \geq 0$ and $(p_2)_{\mathbf{1}} \geq 0$. Then $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$.

(54) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(55) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} > 0$ and $(p_2)_{\mathbf{1}} > 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \geq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(56) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(57) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$. Then $p_1 \leq_P p_2$.

(58) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} \geq 0$ and $(p_2)_{\mathbf{1}} \geq 0$ and $(p_1)_{\mathbf{2}} \geq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(59) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\bf 2} \leq 0$ and $(p_2)_{\bf 2} \leq 0$ and $p_2 \neq {\rm W}_{\rm min}(P)$ and $(p_1)_{\bf 1} \geq (p_2)_{\bf 1}$. Then $p_1 \leq_P p_2$.

(60) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\bf 2} \leq 0$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = c_1$-FanMorphS(q), then $p_{\bf 2} \leq 0$.

(61) Let c_1 be a real number, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$, and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $q_1 = c_1$-FanMorphS(p_1) and $q_2 = c_1$-FanMorphS(p_2). Then $q_1 \leq_P q_2$.

(62) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\bf 1} < 0$ and $(p_1)_{\bf 2} \geq 0$ and $(p_2)_{\bf 1} < 0$ and $(p_2)_{\bf 2} \geq 0$ and $(p_3)_{\bf 1} < 0$ and $(p_3)_{\bf 2} \geq 0$ and $(p_4)_{\bf 1} < 0$ and $(p_4)_{\bf 2} \geq 0$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\bf 1} < 0$ and $(q_1)_{\bf 2} < 0$ and $(q_2)_{\bf 1} < 0$ and $(q_2)_{\bf 2} < 0$ and $(q_3)_{\bf 1} < 0$ and $(q_3)_{\bf 2} < 0$ and $(q_4)_{\bf 1} < 0$ and $(q_4)_{\bf 2} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(63) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\bf 2} \geq 0$ and $(p_2)_{\bf 2} \geq 0$ and $(p_3)_{\bf 2} \geq 0$ and $(p_4)_{\bf 2} > 0$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\bf 1} < 0$ and $(q_1)_{\bf 2} \geq 0$ and $(q_2)_{\bf 1} < 0$ and $(q_2)_{\bf 2} \geq 0$ and $(q_3)_{\bf 1} < 0$ and $(q_3)_{\bf 2} \geq 0$ and $(q_4)_{\bf 1} < 0$ and $(q_4)_{\bf 2} \geq 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(64) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\bf 2} \geq 0$ and $(p_2)_{\bf 2} \geq 0$ and $(p_3)_{\bf 2} \geq 0$ and $(p_4)_{\bf 2} > 0$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\bf 1} < 0$ and $(q_1)_{\bf 2} < 0$ and $(q_2)_{\bf 1} < 0$ and $(q_2)_{\bf 2} < 0$ and $(q_3)_{\bf 1} < 0$ and $(q_3)_{\bf 2} < 0$ and $(q_4)_{\bf 1} < 0$ and $(q_4)_{\bf 2} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(65) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\bf 2} \geq 0$ or $(p_1)_{\bf 1} \geq 0$ and $(p_2)_{\bf 2} \geq 0$ or $(p_2)_{\bf 1} \geq 0$ and $(p_3)_{\bf 2} \geq 0$ or $(p_3)_{\bf 1} \geq 0$ and $(p_4)_{\bf 2} > 0$ or $(p_4)_{\bf 1} > 0$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\bf 2} \geq 0$ and $(q_2)_{\bf 2} \geq 0$ and $(q_3)_{\bf 2} \geq 0$ and $(q_4)_{\bf 2} > 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(66) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\bf 2} \geq 0$ or $(p_1)_{\bf 1} \geq 0$ and $(p_2)_{\bf 2} \geq 0$ or $(p_2)_{\bf 1} \geq 0$ and $(p_3)_{\bf 2} \geq 0$ or $(p_3)_{\bf 1} \geq 0$ and $(p_4)_{\bf 2} > 0$ or $(p_4)_{\bf 1} > 0$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\bf 1} < 0$ and $(q_1)_{\bf 2} < 0$ and $(q_2)_{\bf 1} < 0$ and $(q_2)_{\bf 2} < 0$ and $(q_3)_{\bf 1} < 0$ and $(q_3)_{\bf 2} < 0$ and $(q_4)_{\bf 1} < 0$ and $(q_4)_{\bf 2} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(67) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_4 = {\rm W}_{\rm min}(P)$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(68) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\rm T}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

5. General Fashoda Theorems

We now state several propositions:

(69) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $p_1 \neq p_2$ and $p_2 \neq p_3$ and $p_3 \neq p_4$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_3)_{\mathbf{1}} < 0$ and $(p_4)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_3)_{\mathbf{2}} < 0$ and $(p_4)_{\mathbf{2}} < 0$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $[-1,0] = f(p_1)$ and $[0,1] = f(p_2)$ and $[1,0] = f(p_3)$ and $[0,-1] = f(p_4)$.

(70) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $p_1 \neq p_2$ and $p_2 \neq p_3$ and $p_3 \neq p_4$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ holds $|f(q)| = |q|$ and $[-1,0] = f(p_1)$ and $[0, 1] = f(p_2)$ and $[1,0] = f(p_3)$ and $[0,-1] = f(p_4)$.

(71) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and ${\rm rng}\, f \subseteq C_0$ and ${\rm rng}\, g \subseteq C_0$. Then ${\rm rng}\, f$ meets ${\rm rng}\, g$.

(72) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and ${\rm rng}\, f \subseteq C_0$ and ${\rm rng}\, g \subseteq C_0$. Then ${\rm rng}\, f$ meets ${\rm rng}\, g$.

(73) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and ${\rm rng}\, f \subseteq C_0$ and ${\rm rng}\, g \subseteq C_0$. Then ${\rm rng}\, f$ meets ${\rm rng}\, g$.

(74) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and

$p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

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