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Some Isomorphisms Between Functor Categories

Andrzej Trybulec
Warsaw University
Białystok

Summary. We define some well known isomorphisms between functor categories: between $A^{\dot{\circlearrowleft}(o,m)}$ and A, between $C^{[:A,B:]}$ and $(C^B)^A$, and between $[:B,C:]^A$ and $[:B^A,C^A:]$. Compare [9] and [8]. Unfortunately in this paper "functor" is used in two different meanings, as a lingual function and as a functor between categories.

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The articles [10], [6], [14], [15], [16], [3], [4], [2], [1], [11], [5], [7], [13], and [12] provide the notation and terminology for this paper.

1. Preliminaries

Let A, B, C be non empty sets and let f be a function from A into C^B. Then uncurry f is a function from $[:A, B:]$ into C.

We now state several propositions:

(1) For all non empty sets A, B, C and for every function f from A into C^B holds curry uncurry $f = f$.

(2) Let A, B, C be non empty sets, f be a function from A into C^B, a be an element of A, and b be an element of B. Then $(\text{uncurry } f)(\langle a, b\rangle) = f(a)(b)$.

(3) For every set x and for every non empty set A and for all functions f, g from $\{x\}$ into A such that $f(x) = g(x)$ holds $f = g$.

(4) For all non empty sets A, B and for every element x of A and for every function f from A into B holds $f(x) \in \operatorname{rng} f$.

(5) Let A, B, C be non empty sets and f, g be functions from A into $[:B, C:]$. If $\pi_1(B\times C)\cdot f = \pi_1(B\times C)\cdot g$ and $\pi_2(B\times C)\cdot f = \pi_2(B\times C)\cdot g$, then $f = g$.

In the sequel A, B, C denote categories.

We now state two propositions:

(6) For every morphism f of A holds $\mathrm{id}_{\mathrm{cod}\, f}\cdot f = f$.

(7) For every morphism f of A holds $f\cdot \mathrm{id}_{\mathrm{dom}\, f} = f$.

In the sequel o, m are sets.

The following propositions are true:

(8) o is an object of B^A iff o is a functor from A to B.

(9) Let f be a morphism of B^A. Then there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that F_1 is naturally transformable to F_2 and $\mathrm{dom}\, f = F_1$ and $\mathrm{cod}\, f = F_2$ and $f = \langle\langle F_1, F_2\rangle, t\rangle$.

2. The isomorphism between $A^{\dot{\circlearrowleft}(o,m)}$ and A

Let us consider A, B and let a be an object of A. The functor $a \mapsto B$ yields a functor from B^A to B and is defined by the condition (Def. 1).

(Def. 1) Let F_1, F_2 be functors from A to B and t be a natural transformation from F_1 to F_2. If F_1 is naturally transformable to F_2, then $(a \mapsto B)(\langle\langle F_1, F_2\rangle, t\rangle) = t(a)$.

The following proposition is true

(11)[1] $A^{\dot{\circlearrowleft}(o,m)} \cong A$.

3. The isomorphism between $C^{[:A,B:]}$ and $(C^B)^A$

One can prove the following four propositions:

(12) For every functor F from $[:A, B:]$ to C and for every object a of A and for every object b of B holds $F(a,-)(b) = F(\langle a, b\rangle)$.

(13) For all objects a_1, a_2 of A and for all objects b_1, b_2 of B holds $\mathrm{hom}(a_1,a_2) \neq \emptyset$ and $\mathrm{hom}(b_1,b_2) \neq \emptyset$ iff $\mathrm{hom}(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) \neq \emptyset$.

(14) Let a_1, a_2 be objects of A and b_1, b_2 be objects of B. Suppose $\mathrm{hom}(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) \neq \emptyset$. Let f be a morphism of A and g be a morphism of B. Then $\langle f, g\rangle$ is a morphism from $\langle a_1, b_1\rangle$ to $\langle a_2, b_2\rangle$ if and only if f is a morphism from a_1 to a_2 and g is a morphism from b_1 to b_2.

(15) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2 and a be an object of A. Then $F_1(a,-)$ is naturally transformable to $F_2(a,-)$ and $(\mathrm{curry}\, t)(a)$ is a natural transformation from $F_1(a,-)$ to $F_2(a,-)$.

Let us consider A, B, C, let F be a functor from $[:A, B:]$ to C, and let f be a morphism of A. The functor $\mathrm{curry}(F, f)$ yields a function from the morphisms of B into the morphisms of C and is defined as follows:

(Def. 2) $\mathrm{curry}(F, f) = (\mathrm{curry}\, F)(f)$.

We now state two propositions:

(16) Let a_1, a_2 be objects of A, b_1, b_2 be objects of B, f be a morphism of A, and g be a morphism of B. If $f \in \mathrm{hom}(a_1,a_2)$ and $g \in \mathrm{hom}(b_1,b_2)$, then $\langle f, g\rangle \in \mathrm{hom}(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle)$.

(17) Let F be a functor from $[:A, B:]$ to C and a, b be objects of A. Suppose $\mathrm{hom}(a,b) \neq \emptyset$. Let f be a morphism from a to b. Then

(i) $F(a,-)$ is naturally transformable to $F(b,-)$, and

(ii) $\mathrm{curry}(F, f) \cdot$ the id-map of B is a natural transformation from $F(a,-)$ to $F(b,-)$.

Let us consider A, B, C, let F be a functor from $[:A, B:]$ to C, and let f be a morphism of A. The functor $F(f,-)$ yielding a natural transformation from $F(\mathrm{dom}\, f,-)$ to $F(\mathrm{cod}\, f,-)$ is defined by:

(Def. 3) $F(f,-) = \mathrm{curry}(F, f) \cdot$ the id-map of B.

[1] The proposition (10) has been removed.

One can prove the following propositions:

(18) For every functor F from $[:A, B:]$ to C and for every morphism g of A holds $F(\operatorname{dom} g, -)$ is naturally transformable to $F(\operatorname{cod} g, -)$.

(19) For every functor F from $[:A, B:]$ to C and for every morphism f of A and for every object b of B holds $F(f, -)(b) = F(\langle f, \mathrm{id}_b \rangle)$.

(20) For every functor F from $[:A, B:]$ to C and for every object a of A holds $\mathrm{id}_{F(a,-)} = F(\mathrm{id}_a, -)$.

(21) Let F be a functor from $[:A, B:]$ to C and g, f be morphisms of A. Suppose $\operatorname{dom} g = \operatorname{cod} f$. Let t be a natural transformation from $F(\operatorname{dom} f, -)$ to $F(\operatorname{dom} g, -)$. If $t = F(f, -)$, then $F(g \cdot f, -) = F(g, -) \circ t$.

Let us consider A, B, C and let F be a functor from $[:A, B:]$ to C. The functor $\operatorname{export}(F)$ yields a functor from A to C^B and is defined as follows:

(Def. 4) For every morphism f of A holds $(\operatorname{export}(F))(f) = \langle \langle F(\operatorname{dom} f, -), F(\operatorname{cod} f, -) \rangle, F(f, -) \rangle$.

We now state four propositions:

(24)[2] For every functor F from $[:A, B:]$ to C and for every object a of A holds $(\operatorname{export}(F))(a) = F(a, -)$.

(25) For every functor F from $[:A, B:]$ to C and for every object a of A holds $(\operatorname{export}(F))(a)$ is a functor from B to C.

(26) For all functors F_1, F_2 from $[:A, B:]$ to C such that $\operatorname{export}(F_1) = \operatorname{export}(F_2)$ holds $F_1 = F_2$.

(27) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then

(i) $\operatorname{export}(F_1)$ is naturally transformable to $\operatorname{export}(F_2)$, and

(ii) there exists a natural transformation G from $\operatorname{export}(F_1)$ to $\operatorname{export}(F_2)$ such that for every function s from $[:$the objects of A, the objects of $B:]$ into the morphisms of C such that $t = s$ and for every object a of A holds $G(a) = \langle \langle (\operatorname{export}(F_1))(a), (\operatorname{export}(F_2))(a) \rangle, (\operatorname{curry} s)(a) \rangle$.

Let us consider A, B, C and let F_1, F_2 be functors from $[:A, B:]$ to C. Let us assume that F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. The functor $\operatorname{export}(t)$ yielding a natural transformation from $\operatorname{export}(F_1)$ to $\operatorname{export}(F_2)$ is defined by the condition (Def. 5).

(Def. 5) Let s be a function from $[:$the objects of A, the objects of $B:]$ into the morphisms of C. If $t = s$, then for every object a of A holds $(\operatorname{export}(t))(a) = \langle \langle (\operatorname{export}(F_1))(a), (\operatorname{export}(F_2))(a) \rangle, (\operatorname{curry} s)(a) \rangle$.

We now state several propositions:

(28) For every functor F from $[:A, B:]$ to C holds $\mathrm{id}_{\operatorname{export}(F)} = \operatorname{export}(\mathrm{id}_F)$.

(29) Let F_1, F_2, F_3 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3. Let t_1 be a natural transformation from F_1 to F_2 and t_2 be a natural transformation from F_2 to F_3. Then $\operatorname{export}(t_2 \circ t_1) = \operatorname{export}(t_2) \circ \operatorname{export}(t_1)$.

(30) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t_1, t_2 be natural transformations from F_1 to F_2. If $\operatorname{export}(t_1) = \operatorname{export}(t_2)$, then $t_1 = t_2$.

(31) For every functor G from A to C^B there exists a functor F from $[:A, B:]$ to C such that $G = \operatorname{export}(F)$.

[2] The propositions (22) and (23) have been removed.

(32) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose $\mathrm{export}(F_1)$ is naturally transformable to $\mathrm{export}(F_2)$. Let t be a natural transformation from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$. Then F_1 is naturally transformable to F_2 and there exists a natural transformation u from F_1 to F_2 such that $t = \mathrm{export}(u)$.

Let us consider A, B, C. The functor $\mathbf{export}_{A,B,C}$ yielding a functor from $C^{[:A,B:]}$ to $(C^B)^A$ is defined by the condition (Def. 6).

(Def. 6) Let F_1, F_2 be functors from $[:A, B:]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathbf{export}_{A,B,C}(\langle\langle F_1, F_2\rangle, t\rangle) = \langle\langle \mathrm{export}(F_1), \mathrm{export}(F_2)\rangle, \mathrm{export}(t)\rangle$.

Next we state two propositions:

(33) $\mathbf{export}_{A,B,C}$ is an isomorphism.

(34) $C^{[:A,B:]} \cong (C^B)^A$.

4. THE ISOMORPHISM BETWEEN $[:B,C:]^A$ AND $[:B^A,C^A:]$

We now state the proposition

(35) Let F_1, F_2 be functors from A to B and G be a functor from B to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $G \cdot t = G \cdot (t$ **qua** function$)$.

Let us consider A, B. Then $\pi_1(A \times B)$ is a functor from $[:A, B:]$ to A. Then $\pi_2(A \times B)$ is a functor from $[:A, B:]$ to B.

Let us consider A, B, C, let F be a functor from A to B, and let G be a functor from A to C. Then $\langle F, G\rangle$ is a functor from A to $[:B, C:]$.

Let us consider A, B, C and let F be a functor from A to $[:B, C:]$. The functor $\pi_1 \cdot F$ yields a functor from A to B and is defined by:

(Def. 7) $\pi_1 \cdot F = \pi_1(B \times C) \cdot F$.

The functor $\pi_2 \cdot F$ yields a functor from A to C and is defined by:

(Def. 8) $\pi_2 \cdot F = \pi_2(B \times C) \cdot F$.

Next we state two propositions:

(36) For every functor F from A to B and for every functor G from A to C holds $\pi_1 \cdot \langle F, G\rangle = F$ and $\pi_2 \cdot \langle F, G\rangle = G$.

(37) For all functors F, G from A to $[:B, C:]$ such that $\pi_1 \cdot F = \pi_1 \cdot G$ and $\pi_2 \cdot F = \pi_2 \cdot G$ holds $F = G$.

Let us consider A, B, C, let F_1, F_2 be functors from A to $[:B, C:]$, and let t be a natural transformation from F_1 to F_2. The functor $\pi_1 \cdot t$ yielding a natural transformation from $\pi_1 \cdot F_1$ to $\pi_1 \cdot F_2$ is defined as follows:

(Def. 9) $\pi_1 \cdot t = \pi_1(B \times C) \cdot t$.

The functor $\pi_2 \cdot t$ yields a natural transformation from $\pi_2 \cdot F_1$ to $\pi_2 \cdot F_2$ and is defined as follows:

(Def. 10) $\pi_2 \cdot t = \pi_2(B \times C) \cdot t$.

Next we state several propositions:

(38) Let F, G be functors from A to $[:B, C:]$. Suppose F is naturally transformable to G. Then $\pi_1 \cdot F$ is naturally transformable to $\pi_1 \cdot G$ and $\pi_2 \cdot F$ is naturally transformable to $\pi_2 \cdot G$.

(39) Let F_1, F_2, G_1, G_2 be functors from A to $[:B,C:]$. Suppose F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2. Let s be a natural transformation from F_1 to F_2 and t be a natural transformation from G_1 to G_2. If $\pi_1 \cdot s = \pi_1 \cdot t$ and $\pi_2 \cdot s = \pi_2 \cdot t$, then $s = t$.

(40) For every functor F from A to $[:B,C:]$ holds $\mathrm{id}_{\pi_1 \cdot F} = \pi_1 \cdot (\mathrm{id}_F)$ and $\mathrm{id}_{\pi_2 \cdot F} = \pi_2 \cdot (\mathrm{id}_F)$.

(41) Let F, G, H be functors from A to $[:B,C:]$. Suppose F is naturally transformable to G and G is naturally transformable to H. Let s be a natural transformation from F to G and t be a natural transformation from G to H. Then $\pi_1 \cdot (t \circ s) = \pi_1 \cdot t \circ \pi_1 \cdot s$ and $\pi_2 \cdot (t \circ s) = \pi_2 \cdot t \circ \pi_2 \cdot s$.

(42) Let F be a functor from A to B, G be a functor from A to C, and a, b be objects of A. If $\hom(a,b) \neq \emptyset$, then for every morphism f from a to b holds $\langle F,G \rangle(f) = \langle\!\langle F(f), G(f) \rangle\!\rangle$.

(43) Let F be a functor from A to B, G be a functor from A to C, and a be an object of A. Then $\langle F,G \rangle(a) = \langle\!\langle F(a), G(a) \rangle\!\rangle$.

(44) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is transformable to G_1 and F_2 is transformable to G_2. Then $\langle F_1,F_2 \rangle$ is transformable to $\langle G_1,G_2 \rangle$.

Let us consider A, B, C, let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C. Let us assume that F_1 is transformable to G_1 and F_2 is transformable to G_2. Let t_1 be a transformation from F_1 to G_1 and let t_2 be a transformation from F_2 to G_2. The functor $\langle t_1,t_2 \rangle$ yields a transformation from $\langle F_1,F_2 \rangle$ to $\langle G_1,G_2 \rangle$ and is defined as follows:

(Def. 11) $\langle t_1,t_2 \rangle = \langle t_1,t_2 \rangle$.

Next we state two propositions:

(45) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is transformable to G_1 and F_2 is transformable to G_2. Let t_1 be a transformation from F_1 to G_1, t_2 be a transformation from F_2 to G_2, and a be an object of A. Then $\langle t_1,t_2 \rangle(a) = \langle\!\langle t_1(a), t_2(a) \rangle\!\rangle$.

(46) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Then $\langle F_1,F_2 \rangle$ is naturally transformable to $\langle G_1,G_2 \rangle$.

Let us consider A, B, C, let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C. Let us assume that F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Let t_1 be a natural transformation from F_1 to G_1 and let t_2 be a natural transformation from F_2 to G_2. The functor $\langle t_1,t_2 \rangle$ yields a natural transformation from $\langle F_1,F_2 \rangle$ to $\langle G_1,G_2 \rangle$ and is defined as follows:

(Def. 12) $\langle t_1,t_2 \rangle = \langle t_1,t_2 \rangle$.

One can prove the following proposition

(47) Let F_1, G_1 be functors from A to B and F_2, G_2 be functors from A to C. Suppose F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Let t_1 be a natural transformation from F_1 to G_1 and t_2 be a natural transformation from F_2 to G_2. Then $\pi_1 \cdot \langle t_1,t_2 \rangle = t_1$ and $\pi_2 \cdot \langle t_1,t_2 \rangle = t_2$.

Let us consider A, B, C. The functor $\mathbf{distribute}_{A,B,C}$ yields a functor from $[:B,C:]^A$ to $[:B^A,C^A:]$ and is defined by the condition (Def. 13).

(Def. 13) Let F_1, F_2 be functors from A to $[:B,C:]$. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathbf{distribute}_{A,B,C}(\langle\!\langle \langle\!\langle F_1, F_2 \rangle\!\rangle, t \rangle\!\rangle) = \langle\!\langle \langle\!\langle \langle\!\langle \pi_1 \cdot F_1, \pi_1 \cdot F_2 \rangle\!\rangle, \pi_1 \cdot t \rangle\!\rangle, \langle\!\langle \langle\!\langle \pi_2 \cdot F_1, \pi_2 \cdot F_2 \rangle\!\rangle, \pi_2 \cdot t \rangle\!\rangle \rangle\!\rangle$.

One can prove the following two propositions:

(48) $\mathbf{distribute}_{A,B,C}$ is an isomorphism.

(49) $[:B,C:]^A \cong [:B^A,C^A:]$.

REFERENCES

[1] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_5.html.

[2] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_3.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Czesław Byliński. Introduction to categories and functors. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/cat_1.html.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[7] Czesław Byliński. Subcategories and products of categories. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/cat_2.html.

[8] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer Verlag, New York, Heidelberg, Berlin, 1971.

[9] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów*, volume 45 of *Biblioteka Matematyczna*. PWN, Warszawa, 1978.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[12] Andrzej Trybulec. Isomorphisms of categories. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/isocat_1.html.

[13] Andrzej Trybulec. Natural transformations. Discrete categories. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/nattra_1.html.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[16] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

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