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The Ring of Integers, Euclidean Rings and Modulo Integers

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Summary. In this article we introduce the ring of Integers, Euclidean rings and Integers modulo p. In particular we prove that the Ring of Integers is an Euclidean ring and that the Integers modulo p constitutes a field if and only if p is a prime.

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The articles [15], [20], [2], [16], [18], [21], [5], [6], [13], [10], [12], [4], [9], [8], [14], [17], [3], [7], [11], [1], and [19] provide the notation and terminology for this paper.

1. The Ring of Integers

The binary operation multint on $\mathbb{Z}$ is defined by:

(Def. 1) For all elements a, b of $\mathbb{Z}$ holds $(\text{multint})(a,b) = \cdot_{\mathbb{R}}(a,b)$.

The unary operation compint on $\mathbb{Z}$ is defined by:

(Def. 2) For every element a of $\mathbb{Z}$ holds $(\text{compint})(a) = -_{\mathbb{R}}(a)$.

The double loop structure INT.Ring is defined as follows:

(Def. 3) $\text{INT.Ring} = \langle \mathbb{Z}, +_{\mathbb{Z}}, \text{multint}, 1(\in \mathbb{Z}), 0(\in \mathbb{Z}) \rangle$.

One can check that INT.Ring is strict and non empty.

One can check that INT.Ring is Abelian, add-associative, right zeroed, right complementable, well unital, distributive, commutative, associative, integral domain-like, and non degenerated.

Let a, b be elements of INT.Ring. The predicate $a \leq b$ is defined by:

(Def. 4) There exist integers a', b' such that $a' = a$ and $b' = b$ and $a' \leq b'$.

Let us notice that the predicate $a \leq b$ is reflexive and connected. We introduce $b \geq a$ as a synonym of $a \leq b$. We introduce $b < a$ and $a > b$ as antonyms of $a \leq b$.

Let a be an element of INT.Ring. The functor $|a|$ yields an element of INT.Ring and is defined by:

(Def. 5) $|a| = \begin{cases} a, \text{ if } a \geq 0_{\text{INT.Ring}}, \\ -a, \text{ otherwise.} \end{cases}$

The function absint from the carrier of INT.Ring into $\mathbb{N}$ is defined by:

(Def. 6) For every element a of INT.Ring holds $(\text{absint})(a) = |\Box|_{\mathbb{R}}(a)$.

Next we state two propositions:

(1) For every element a of INT.Ring holds $(\mathrm{absint})(a) = |a|$.

(2) Let a, b, q_1, q_2, r_1, r_2 be elements of INT.Ring. Suppose $b \neq 0_{\mathrm{INT.Ring}}$ and $a = q_1 \cdot b + r_1$ and $0_{\mathrm{INT.Ring}} \leq r_1$ and $r_1 < |b|$ and $a = q_2 \cdot b + r_2$ and $0_{\mathrm{INT.Ring}} \leq r_2$ and $r_2 < |b|$. Then $q_1 = q_2$ and $r_1 = r_2$.

Let a, b be elements of INT.Ring. Let us assume that $b \neq 0_{\mathrm{INT.Ring}}$. The functor $a \div b$ yields an element of INT.Ring and is defined by:

(Def. 7) There exists an element r of INT.Ring such that $a = (a \div b) \cdot b + r$ and $0_{\mathrm{INT.Ring}} \leq r$ and $r < |b|$.

Let a, b be elements of INT.Ring. Let us assume that $b \neq 0_{\mathrm{INT.Ring}}$. The functor $a \bmod b$ yielding an element of INT.Ring is defined by:

(Def. 8) There exists an element q of INT.Ring such that $a = q \cdot b + (a \bmod b)$ and $0_{\mathrm{INT.Ring}} \leq a \bmod b$ and $a \bmod b < |b|$.

We now state the proposition

(3) For all elements a, b of INT.Ring such that $b \neq 0_{\mathrm{INT.Ring}}$ holds $a = (a \div b) \cdot b + (a \bmod b)$.

2. Euclidean Rings

Let I be a non empty double loop structure. We say that I is Euclidian if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exists a function f from the carrier of I into $\mathbb{N}$ such that for all elements a, b of I if $b \neq 0_I$, then there exist elements q, r of I such that $a = q \cdot b + r$ but $r = 0_I$ or $f(r) < f(b)$.

One can verify that INT.Ring is Euclidian.

One can check that there exists a ring which is strict, Euclidian, integral domain-like, non degenerated, well unital, distributive, and commutative.

An EuclidianRing is an Euclidian integral domain-like non degenerated well unital distributive commutative ring.

Let us note that there exists an EuclidianRing which is strict.

Let E be an Euclidian non empty double loop structure. A function from the carrier of E into $\mathbb{N}$ is said to be a DegreeFunction of E if:

(Def. 10) For all elements a, b of E such that $b \neq 0_E$ there exist elements q, r of E such that $a = q \cdot b + r$ but $r = 0_E$ or $\mathrm{it}(r) < \mathrm{it}(b)$.

Next we state the proposition

(4) Every EuclidianRing is a gcdDomain.

One can verify that every integral domain-like non degenerated Abelian add-associative right zeroed right complementable associative commutative right unital right distributive non empty double loop structure which is Euclidian is also gcd-like.

absint is a DegreeFunction of INT.Ring.

One can prove the following proposition

(5) Every commutative associative left unital field-like right zeroed non empty double loop structure is Euclidian.

Let us note that every non empty double loop structure which is commutative, associative, left unital, field-like, right zeroed, and field-like is also Euclidian.

One can prove the following proposition

(6) Let F be a commutative associative left unital field-like right zeroed non empty double loop structure. Then every function from the carrier of F into $\mathbb{N}$ is a DegreeFunction of F.

3. Some Theorems about Div and Mod

Next we state several propositions:

(8)[1] For every natural number n such that $n > 0$ and for all integers a, k holds $(a + n \cdot k) \div n = (a \div n) + k$ and $(a + n \cdot k) \bmod n = a \bmod n$.

(9) For every natural number n such that $n > 0$ and for every integer a holds $a \bmod n \geq 0$ and $a \bmod n < n$.

(10) Let n be a natural number. Suppose $n > 0$. Let a be an integer. Then

(i) if $0 \leq a$ and $a < n$, then $a \bmod n = a$, and

(ii) if $0 > a$ and $a \geq -n$, then $a \bmod n = n + a$.

(11) For every natural number n such that $n > 0$ and for every integer a holds $a \bmod n = 0$ iff $n \mid a$.

(12) For every natural number n such that $n > 0$ and for all integers a, b holds $a \bmod n = b \bmod n$ iff $a \equiv b \pmod n$.

(13) For every natural number n such that $n > 0$ and for every integer a holds $a \bmod n \bmod n = a \bmod n$.

(14) For every natural number n such that $n > 0$ and for all integers a, b holds $(a + b) \bmod n = ((a \bmod n) + (b \bmod n)) \bmod n$.

(15) For every natural number n such that $n > 0$ and for all integers a, b holds $a \cdot b \bmod n = (a \bmod n) \cdot (b \bmod n) \bmod n$.

(16) For all integers a, b there exist integers s, t such that $a \gcd b = s \cdot a + t \cdot b$.

4. Modulo Integers

Let n be a natural number. Let us assume that $n > 0$. The functor multint$n$ yields a binary operation on $\mathbb{Z}_n$ and is defined as follows:

(Def. 11) For all elements k, l of $\mathbb{Z}_n$ holds $(\text{multint}\, n)(k, l) = k \cdot l \bmod n$.

Let n be a natural number. Let us assume that $n > 0$. The functor compint$n$ yields a unary operation on $\mathbb{Z}_n$ and is defined as follows:

(Def. 12) For every element k of $\mathbb{Z}_n$ holds $(\text{compint}\, n)(k) = (n - k) \bmod n$.

The following three propositions are true:

(17) Let n be a natural number. Suppose $n > 0$. Let a, b be elements of $\mathbb{Z}_n$. Then

(i) $a + b < n$ iff $+_n(a, b) = a + b$, and

(ii) $a + b \geq n$ iff $+_n(a, b) = (a + b) - n$.

(18) Let n be a natural number. Suppose $n > 0$. Let a, b be elements of $\mathbb{Z}_n$ and k be a natural number. Then $k \cdot n \leq a \cdot b$ and $a \cdot b < (k + 1) \cdot n$ if and only if $(\text{multint}\, n)(a, b) = a \cdot b - k \cdot n$.

(19) Let n be a natural number. Suppose $n > 0$. Let a be an element of $\mathbb{Z}_n$. Then

(i) $a = 0$ iff $(\text{compint}\, n)(a) = 0$, and

(ii) $a \neq 0$ iff $(\text{compint}\, n)(a) = n - a$.

Let n be a natural number. The functor INT.Ring$n$ yielding a double loop structure is defined by:

[1] The proposition (7) has been removed.

(Def. 13) INT.Ring $n = \langle \mathbb{Z}_n, +_n, \text{multint}\, n, 1(\in \mathbb{Z}_n), 0(\in \mathbb{Z}_n) \rangle$.

Let n be a natural number. One can check that INT.Ring n is strict and non empty.

Next we state the proposition

(20) INT.Ring 1 is degenerated and INT.Ring 1 is a ring and INT.Ring 1 is field-like, well unital, distributive, and commutative.

One can verify that there exists a ring which is strict, degenerated, well unital, distributive, field-like, and commutative.

Next we state two propositions:

(21) Let n be a natural number. Suppose $n > 1$. Then INT.Ring n is non degenerated and INT.Ring n is a well unital distributive commutative ring.

(22) Let p be a natural number. Suppose $p > 1$. Then INT.Ring p is an add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure if and only if p is a prime number.

Let p be a prime number. Note that INT.Ring p is add-associative, right zeroed, right complementable, Abelian, commutative, associative, left unital, distributive, field-like, and non degenerated.

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