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Indexed Category

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Summary. The concept of indexing of a category (a part of indexed category, see [14]) is introduced as a pair formed by a many sorted category and a many sorted functor. The indexing of a category C against to [14] is not a functor but it can be treated as a functor from C into some categorial category (see [1]). The goal of the article is to work out the notation necessary to define institutions (see [11]).

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The articles [15], [8], [20], [16], [21], [4], [5], [7], [18], [17], [19], [12], [3], [6], [9], [10], [2], [13], and [1] provide the notation and terminology for this paper.

1. Category-yielding Functions

Let A be a non empty set. One can verify that there exists a many sorted set indexed by A which is non empty yielding.

Let A be a non empty set. One can verify that every many sorted set indexed by A which is non-empty is also non empty yielding.

Let C be a categorial category and let f be a morphism of C. Then $f_{\mathbf{2}}$ is a functor from $f_{\mathbf{1},\mathbf{1}}$ to $f_{\mathbf{1},\mathbf{2}}$.

One can prove the following propositions:

(1) For every categorial category C and for all morphisms f, g of C such that $\operatorname{dom} g = \operatorname{cod} f$ holds $g \cdot f = \langle\langle \operatorname{dom} f, \operatorname{cod} g\rangle, g_{\mathbf{2}} \cdot f_{\mathbf{2}}\rangle$.

(2) Let C be a category, D, E be categorial categories, F be a functor from C to D, and G be a functor from C to E. If $F = G$, then $\operatorname{Obj} F = \operatorname{Obj} G$.

Let I_1 be a function. We say that I_1 is category-yielding if and only if:

(Def. 1) For every set x such that $x \in \operatorname{dom} I_1$ holds $I_1(x)$ is a category.

Let us note that there exists a function which is category-yielding.

Let X be a set. One can verify that there exists a many sorted set indexed by X which is category-yielding.

Let A be a set. A many sorted category indexed by A is a category-yielding many sorted set indexed by A.

Let C be a category. A many sorted set indexed by C is a many sorted set indexed by the objects of C. A many sorted category indexed by C is a many sorted category indexed by the objects of C.

Let X be a set and let x be a category. Observe that $X \longmapsto x$ is category-yielding.

Let X be a non empty set. Note that every many sorted set indexed by X is non empty.

Let f be a category-yielding function. Observe that $\operatorname{rng} f$ is categorial.

Let X be a non empty set, let f be a many sorted category indexed by X, and let x be an element of X. Then $f(x)$ is a category.

Let f be a function and let g be a category-yielding function. One can verify that $g \cdot f$ is category-yielding.

Let F be a category-yielding function. The functor $\operatorname{Objs}(F)$ yielding a non-empty function is defined by the conditions (Def. 2).

(Def. 2)(i) $\operatorname{dom}\operatorname{Objs}(F) = \operatorname{dom} F$, and

(ii) for every set x such that $x \in \operatorname{dom} F$ and for every category C such that $C = F(x)$ holds $(\operatorname{Objs}(F))(x) =$ the objects of C.

The functor $\operatorname{Mphs}(F)$ yields a non-empty function and is defined by the conditions (Def. 3).

(Def. 3)(i) $\operatorname{dom}\operatorname{Mphs}(F) = \operatorname{dom} F$, and

(ii) for every set x such that $x \in \operatorname{dom} F$ and for every category C such that $C = F(x)$ holds $(\operatorname{Mphs}(F))(x) =$ the morphisms of C.

Let A be a non empty set and let F be a many sorted category indexed by A. Then $\operatorname{Objs}(F)$ is a non-empty many sorted set indexed by A. Then $\operatorname{Mphs}(F)$ is a non-empty many sorted set indexed by A.

Next we state the proposition

(3) For every set X and for every category C holds $\operatorname{Objs}(X \longmapsto C) = X \longmapsto$ the objects of C and $\operatorname{Mphs}(X \longmapsto C) = X \longmapsto$ the morphisms of C.

2. Pairs of Many Sorted Sets

Let A, B be sets. Pair of many sorted sets indexed by A and B is defined by:

(Def. 4) There exists a many sorted set f indexed by A and there exists a many sorted set g indexed by B such that it $= \langle f, g \rangle$.

Let A, B be sets, let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. Then $\langle f, g \rangle$ is a pair of many sorted sets indexed by A and B.

Let A, B be sets and let X be a pair of many sorted sets indexed by A and B. Then $X_{\mathbf{1}}$ is a many sorted set indexed by A. Then $X_{\mathbf{2}}$ is a many sorted set indexed by B.

Let A, B be sets and let I_1 be a pair of many sorted sets indexed by A and B. We say that I_1 is category-yielding on first if and only if:

(Def. 5) $(I_1)_{\mathbf{1}}$ is category-yielding.

We say that I_1 is function-yielding on second if and only if:

(Def. 6) $(I_1)_{\mathbf{2}}$ is function yielding.

Let A, B be sets. Note that there exists a pair of many sorted sets indexed by A and B which is category-yielding on first and function-yielding on second.

Let A, B be sets and let X be a category-yielding on first pair of many sorted sets indexed by A and B. Then $X_{\mathbf{1}}$ is a many sorted category indexed by A.

Let A, B be sets and let X be a function-yielding on second pair of many sorted sets indexed by A and B. Then $X_{\mathbf{2}}$ is a many sorted function indexed by B.

Let f be a function yielding function. One can verify that $\operatorname{rng} f$ is functional.

Let A, B be sets, let f be a many sorted category indexed by A, and let g be a many sorted function indexed by B. Then $\langle f, g \rangle$ is a category-yielding on first function-yielding on second pair of many sorted sets indexed by A and B.

Let A be a non empty set and let F, G be many sorted categories indexed by A. A many sorted function indexed by A is said to be a many sorted functor from F to G if:

(Def. 7) For every element a of A holds $\mathrm{it}(a)$ is a functor from $F(a)$ to $G(a)$.

The scheme *LambdaMSFr* deals with a non empty set $\mathcal{A}$, many sorted categories $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a many sorted functor F from $\mathcal{B}$ to $\mathcal{C}$ such that for every element a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$

provided the parameters meet the following condition:

- For every element a of $\mathcal{A}$ holds $\mathcal{F}(a)$ is a functor from $\mathcal{B}(a)$ to $\mathcal{C}(a)$.

Let A be a non empty set, let F, G be many sorted categories indexed by A, let f be a many sorted functor from F to G, and let a be an element of A. Then $f(a)$ is a functor from $F(a)$ to $G(a)$.

3. INDEXING

Let A, B be non empty sets and let F, G be functions from B into A. A category-yielding on first pair of many sorted sets indexed by A and B is said to be an indexing of F and G if:

(Def. 8) $\mathrm{it}_{\mathbf{2}}$ is a many sorted functor from $\mathrm{it}_{\mathbf{1}} \cdot F$ to $\mathrm{it}_{\mathbf{1}} \cdot G$.

The following two propositions are true:

(4) Let A, B be non empty sets, F, G be functions from B into A, I be an indexing of F and G, and m be an element of B. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(F(m))$ to $I_{\mathbf{1}}(G(m))$.

(5) Let C be a category, I be an indexing of the dom-map of C and the cod-map of C, and m be a morphism of C. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(\mathrm{dom}\, m)$ to $I_{\mathbf{1}}(\mathrm{cod}\, m)$.

Let A, B be non empty sets, let F, G be functions from B into A, and let I be an indexing of F and G. Then $I_{\mathbf{2}}$ is a many sorted functor from $I_{\mathbf{1}} \cdot F$ to $I_{\mathbf{1}} \cdot G$.

Let A, B be non empty sets, let F, G be functions from B into A, and let I be an indexing of F and G. A categorial category is said to be a target category of I if it satisfies the conditions (Def. 9).

(Def. 9)(i) For every element a of A holds $I_{\mathbf{1}}(a)$ is an object of it, and

(ii) for every element b of B holds $\langle\langle I_{\mathbf{1}}(F(b)), I_{\mathbf{1}}(G(b))\rangle, I_{\mathbf{2}}(b)\rangle$ is a morphism of it.

Let A, B be non empty sets, let F, G be functions from B into A, and let I be an indexing of F and G. Note that there exists a target category of I which is full and strict.

Let A, B be non empty sets, let F, G be functions from B into A, let c be a partial function from $[:B, B:]$ to B, and let i be a function from A into B. Let us assume that there exists a category C such that $C = \langle A, B, F, G, c, i\rangle$. An indexing of F and G is said to be an indexing of F, G, c and i if it satisfies the conditions (Def. 10).

(Def. 10)(i) For every element a of A holds $\mathrm{it}_{\mathbf{2}}(i(a)) = \mathrm{id}_{\mathrm{it}_{\mathbf{1}}(a)}$, and

(ii) for all elements m_1, m_2 of B such that $F(m_2) = G(m_1)$ holds $\mathrm{it}_{\mathbf{2}}(c(\langle m_2, m_1\rangle)) = \mathrm{it}_{\mathbf{2}}(m_2) \cdot \mathrm{it}_{\mathbf{2}}(m_1)$.

Let C be a category. An indexing of C is an indexing of the dom-map of C, the cod-map of C, the composition of C and the id-map of C. A coindexing of C is an indexing of the cod-map of C, the dom-map of C, $\curvearrowleft$(the composition of C) and the id-map of C.

One can prove the following propositions:

(6) Let C be a category and I be an indexing of the dom-map of C and the cod-map of C. Then I is an indexing of C if and only if the following conditions are satisfied:

(i) for every object a of C holds $I_{\mathbf{2}}(\mathrm{id}_a) = \mathrm{id}_{I_{\mathbf{1}}(a)}$, and

(ii) for all morphisms m_1, m_2 of C such that $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ holds $I_{\mathbf{2}}(m_2 \cdot m_1) = I_{\mathbf{2}}(m_2) \cdot I_{\mathbf{2}}(m_1)$.

(7) Let C be a category and I be an indexing of the cod-map of C and the dom-map of C. Then I is a coindexing of C if and only if the following conditions are satisfied:

(i) for every object a of C holds $I_{\mathbf{2}}(\mathrm{id}_a) = \mathrm{id}_{I_{\mathbf{1}}(a)}$, and

(ii) for all morphisms m_1, m_2 of C such that $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ holds $I_{\mathbf{2}}(m_2 \cdot m_1) = I_{\mathbf{2}}(m_1) \cdot I_{\mathbf{2}}(m_2)$.

(8) For every category C and for every set x holds x is a coindexing of C iff x is an indexing of C^{op}.

(9) Let C be a category, I be an indexing of C, and c_1, c_2 be objects of C. Suppose $\hom(c_1, c_2)$ is non empty. Let m be a morphism from c_1 to c_2. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(c_1)$ to $I_{\mathbf{1}}(c_2)$.

(10) Let C be a category, I be a coindexing of C, and c_1, c_2 be objects of C. Suppose $\hom(c_1, c_2)$ is non empty. Let m be a morphism from c_1 to c_2. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(c_2)$ to $I_{\mathbf{1}}(c_1)$.

Let C be a category, let I be an indexing of C, and let T be a target category of I. The functor I-functor(C, T) yielding a functor from C to T is defined by:

(Def. 11) For every morphism f of C holds $(I\text{-functor}(C, T))(f) = \langle\langle I_{\mathbf{1}}(\mathrm{dom}\, f), I_{\mathbf{1}}(\mathrm{cod}\, f)\rangle, I_{\mathbf{2}}(f)\rangle$.

The following propositions are true:

(11) Let C be a category, I be an indexing of C, and T_1, T_2 be target categories of I. Then $I\text{-functor}(C, T_1) = I\text{-functor}(C, T_2)$ and $\mathrm{Obj}(I\text{-functor}(C, T_1)) = \mathrm{Obj}(I\text{-functor}(C, T_2))$.

(12) For every category C and for every indexing I of C and for every target category T of I holds $\mathrm{Obj}(I\text{-functor}(C, T)) = I_{\mathbf{1}}$.

(13) Let C be a category, I be an indexing of C, T be a target category of I, and x be an object of C. Then $(I\text{-functor}(C, T))(x) = I_{\mathbf{1}}(x)$.

Let C be a category and let I be an indexing of C. The functor $\mathrm{rng}\, I$ yields a strict target category of I and is defined as follows:

(Def. 12) For every target category T of I holds $\mathrm{rng}\, I = \mathrm{Im}(I\text{-functor}(C, T))$.

The following proposition is true

(14) Let C be a category, I be an indexing of C, and D be a categorial category. Then $\mathrm{rng}\, I$ is a subcategory of D if and only if D is a target category of I.

Let C be a category, let I be an indexing of C, and let m be a morphism of C. The functor $I(m)$ yielding a functor from $I_{\mathbf{1}}(\mathrm{dom}\, m)$ to $I_{\mathbf{1}}(\mathrm{cod}\, m)$ is defined as follows:

(Def. 13) $I(m) = I_{\mathbf{2}}(m)$.

Let C be a category, let I be a coindexing of C, and let m be a morphism of C. The functor $I(m)$ yielding a functor from $I_{\mathbf{1}}(\mathrm{cod}\, m)$ to $I_{\mathbf{1}}(\mathrm{dom}\, m)$ is defined by:

(Def. 14) $I(m) = I_{\mathbf{2}}(m)$.

The following proposition is true

(15) Let C, D be categories. Then

(i) $\langle$(the objects of C) $\longmapsto D$, (the morphisms of C) $\longmapsto \mathrm{id}_D\rangle$ is an indexing of C, and

(ii) $\langle$(the objects of C) $\longmapsto D$, (the morphisms of C) $\longmapsto \mathrm{id}_D\rangle$ is a coindexing of C.

4. Indexing vs Functors

Let C be a category, let D be a categorial category, and let F be a functor from C to D. One can verify that $\mathrm{Obj}\,F$ is category-yielding.

Next we state the proposition

(16) Let C be a category, D be a categorial category, and F be a functor from C to D. Then $\langle \mathrm{Obj}\,F, \mathrm{pr2}(F)\rangle$ is an indexing of C.

Let C be a category, let D be a categorial category, and let F be a functor from C to D. The functor F-indexing of C yields an indexing of C and is defined by:

(Def. 15) F-indexing of $C = \langle \mathrm{Obj}\,F, \mathrm{pr2}(F)\rangle$.

Next we state several propositions:

(17) Let C be a category, D be a categorial category, and F be a functor from C to D. Then D is a target category of F-indexing of C.

(18) Let C be a category, D be a categorial category, F be a functor from C to D, and T be a target category of F-indexing of C. Then $F = F$-indexing of C-functor(C,T).

(19) Let C be a category, D, E be categorial categories, F be a functor from C to D, and G be a functor from C to E. If $F = G$, then F-indexing of $C = G$-indexing of C.

(20) For every category C and for every indexing I of C and for every target category T of I holds $\mathrm{pr2}(I\text{-functor}(C,T)) = I_{\mathbf{2}}$.

(21) For every category C and for every indexing I of C and for every target category T of I holds $(I\text{-functor}(C,T))$-indexing of $C = I$.

5. Composing Indexings and Functors

Let C, D, E be categories, let F be a functor from C to D, and let I be an indexing of E. Let us assume that $\mathrm{Im}\,F$ is a subcategory of E. The functor $I \cdot F$ yields an indexing of C and is defined by:

(Def. 16) For every functor F' from C to E such that $F' = F$ holds $I \cdot F = ((I\text{-functor}(E, \mathrm{rng}\,I)) \cdot F')$-indexing of C.

Next we state several propositions:

(22) Let C, D_1, D_2, E be categories, I be an indexing of E, F be a functor from C to D_1, and G be a functor from C to D_2. Suppose $\mathrm{Im}\,F$ is a subcategory of E and $\mathrm{Im}\,G$ is a subcategory of E and $F = G$. Then $I \cdot F = I \cdot G$.

(23) Let C, D be categories, F be a functor from C to D, I be an indexing of D, and T be a target category of I. Then $I \cdot F = ((I\text{-functor}(D,T)) \cdot F)$-indexing of C.

(24) Let C, D be categories, F be a functor from C to D, and I be an indexing of D. Then every target category of I is a target category of $I \cdot F$.

(25) Let C, D be categories, F be a functor from C to D, I be an indexing of D, and T be a target category of I. Then $\mathrm{rng}(I \cdot F)$ is a subcategory of T.

(26) Let C, D, E be categories, F be a functor from C to D, G be a functor from D to E, and I be an indexing of E. Then $(I \cdot G) \cdot F = I \cdot (G \cdot F)$.

Let C be a category, let I be an indexing of C, and let D be a categorial category. Let us assume that D is a target category of I. Let E be a categorial category and let F be a functor from D to E. The functor $F \cdot I$ yields an indexing of C and is defined as follows:

(Def. 17) For every target category T of I and for every functor G from T to E such that $T = D$ and $G = F$ holds $F \cdot I = (G \cdot (I\text{-functor}(C,T)))$-indexing of C.

Next we state several propositions:

(27) Let C be a category, I be an indexing of C, T be a target category of I, D, E be categorial categories, F be a functor from T to D, and G be a functor from T to E. If $F = G$, then $F \cdot I = G \cdot I$.

(28) Let C be a category, I be an indexing of C, T be a target category of I, D be a categorial category, and F be a functor from T to D. Then $\mathrm{Im} F$ is a target category of $F \cdot I$.

(29) Let C be a category, I be an indexing of C, T be a target category of I, D be a categorial category, and F be a functor from T to D. Then D is a target category of $F \cdot I$.

(30) Let C be a category, I be an indexing of C, T be a target category of I, D be a categorial category, and F be a functor from T to D. Then $\mathrm{rng}(F \cdot I)$ is a subcategory of $\mathrm{Im} F$.

(31) Let C be a category, I be an indexing of C, T be a target category of I, D, E be categorial categories, F be a functor from T to D, and G be a functor from D to E. Then $(G \cdot F) \cdot I = G \cdot (F \cdot I)$.

Let C, D be categories, let I_2 be an indexing of C, and let I_3 be an indexing of D. The functor $I_3 \cdot I_2$ yields an indexing of C and is defined by:

(Def. 18) $I_3 \cdot I_2 = I_3 \cdot (I_2\text{-functor}(C, \mathrm{rng} I_2))$.

Next we state several propositions:

(32) Let C be a category, D be a categorial category, I_2 be an indexing of C, I_3 be an indexing of D, and T be a target category of I_2. If D is a target category of I_2, then $I_3 \cdot I_2 = I_3 \cdot (I_2\text{-functor}(C,T))$.

(33) Let C be a category, D be a categorial category, I_2 be an indexing of C, I_3 be an indexing of D, and T be a target category of I_3. If D is a target category of I_2, then $I_3 \cdot I_2 = (I_3\text{-functor}(D,T)) \cdot I_2$.

(34) Let C, D be categories, F be a functor from C to D, I be an indexing of D, T be a target category of I, E be a categorial category, and G be a functor from T to E. Then $(G \cdot I) \cdot F = G \cdot (I \cdot F)$.

(35) Let C be a category, I be an indexing of C, T be a target category of I, D be a categorial category, F be a functor from T to D, and J be an indexing of D. Then $(J \cdot F) \cdot I = J \cdot (F \cdot I)$.

(36) Let C be a category, I be an indexing of C, T_1 be a target category of I, J be an indexing of T_1, T_2 be a target category of J, D be a categorial category, and F be a functor from T_2 to D. Then $(F \cdot J) \cdot I = F \cdot (J \cdot I)$.

(37) Let C, D be categories, F be a functor from C to D, I be an indexing of D, T be a target category of I, and J be an indexing of T. Then $(J \cdot I) \cdot F = J \cdot (I \cdot F)$.

(38) Let C be a category, I be an indexing of C, D be a target category of I, J be an indexing of D, E be a target category of J, and K be an indexing of E. Then $(K \cdot J) \cdot I = K \cdot (J \cdot I)$.

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