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Incidence Projective Spaces

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Summary. A basis for investigations on incidence projective spaces. With every projective space defined in terms of collinearity relation we associate the incidence structure consisting of points and of lines of the given space. We introduce general notion of projective space defined in terms of incidence and define several properties of such structures (like satisfability of the Desargues Axiom or conditions on the dimension).

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The articles [4], [1], [6], [7], [5], [3], and [2] provide the notation and terminology for this paper.

We adopt the following convention: C_1 denotes a proper collinearity space, B denotes an element of $2^{\text{the carrier of } C_1}$, and x, y, Y denote sets.

Let us consider C_1. We see that the line of C_1 is an element of $2^{\text{the carrier of } C_1}$.

Let us consider C_1. The functor $L(C_1)$ yields a set and is defined by:

(Def. 1) $L(C_1) = \{B : B \text{ is a line of } C_1\}$.

Let us consider C_1. Observe that $L(C_1)$ is non empty.

The following proposition is true

(2)[1] x is a line of C_1 iff x is an element of $L(C_1)$.

Let us consider C_1. The functor $\mathbf{I}_{(C_1)}$ yielding a relation between the carrier of C_1 and $L(C_1)$ is defined as follows:

(Def. 2) For all x, y holds $\langle x, y\rangle \in \mathbf{I}_{(C_1)}$ iff $x \in$ the carrier of C_1 and $y \in L(C_1)$ and there exists Y such that $y = Y$ and $x \in Y$.

Let us consider C_1. The functor Inc-ProjSp(C_1) yields a projective incidence structure and is defined as follows:

(Def. 3) Inc-ProjSp$(C_1) = \langle$the carrier of C_1, $L(C_1), \mathbf{I}_{(C_1)}\rangle$.

Let us consider C_1. One can check that Inc-ProjSp(C_1) is strict.

We now state three propositions:

(4)[2] The points of Inc-ProjSp(C_1) = the carrier of C_1 and the lines of Inc-ProjSp$(C_1) = L(C_1)$ and the incidence of Inc-ProjSp$(C_1) = \mathbf{I}_{(C_1)}$.

(5) x is a point of C_1 iff x is a point of Inc-ProjSp(C_1).

[1] The proposition (1) has been removed.
[2] The proposition (3) has been removed.

(6) x is a line of C_1 iff x is a line of Inc-ProjSp(C_1).

For simplicity, we follow the rules: a, b, c, p, q, s are points of Inc-ProjSp(C_1), P, Q, S are lines of Inc-ProjSp(C_1), a', b', c', p' are points of C_1, and P' is a line of C_1.

Next we state several propositions:

(8)[3] s lies on S iff $\langle s, S\rangle \in \mathbf{I}_{(C_1)}$.

(9) If $p = p'$ and $P = P'$, then p lies on P iff $p' \in P'$.

(10) There exist a', b', c' such that $a' \neq b'$ and $b' \neq c'$ and $c' \neq a'$.

(11) There exists b' such that $a' \neq b'$.

(12) If p lies on P and q lies on P and p lies on Q and q lies on Q, then $p = q$ or $P = Q$.

(13) There exists P such that p lies on P and q lies on P.

(14) Suppose $a = a'$ and $b = b'$ and $c = c'$. Then a', b' and c' are collinear if and only if there exists P such that a lies on P and b lies on P and c lies on P.

(15) There exist p, P such that p does not lie on P.

For simplicity, we adopt the following convention: C_1 is a projective space defined in terms of collinearity, a, b, c, d, p, q are points of Inc-ProjSp(C_1), P, Q, S, M, N are lines of Inc-ProjSp(C_1), and a', b', c', d', p' are points of C_1.

The following propositions are true:

(16) There exist a, b, c such that $a \neq b$ and $b \neq c$ and $c \neq a$ and a lies on P and b lies on P and c lies on P.

(17) Suppose that a lies on M and b lies on M and c lies on N and d lies on N and p lies on M and p lies on N and a lies on P and c lies on P and b lies on Q and d lies on Q and p does not lie on P and p does not lie on Q and $M \neq N$. Then there exists q such that q lies on P and q lies on Q.

(18) Suppose that for all a', b', c', d' there exists p' such that a', b' and p' are collinear and c', d' and p' are collinear. Let given M, N. Then there exists q such that q lies on M and q lies on N.

(19) Suppose that it is not true that there exist points p, p_1, r, r_1 of C_1 and there exists a point s of C_1 such that p, p_1 and s are collinear and r, r_1 and s are collinear. Then it is not true that there exist M, N and there exists q such that q lies on M and q lies on N.

(20) Suppose that for all points p, p_1, q, q_1, r_2 of C_1 there exist points r, r_1 of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear. Let given a, M, N. Then there exist b, c, S such that a lies on S and b lies on S and c lies on S and b lies on M and c lies on N.

Let x, y, z be sets. We say that x, y, z are mutually different if and only if:

(Def. 5)[4] $x \neq y$ and $y \neq z$ and $z \neq x$.

Let u be a set. We say that x, y, z, u are mutually different if and only if:

(Def. 6) $x \neq y$ and $y \neq z$ and $z \neq x$ and $u \neq x$ and $u \neq y$ and $u \neq z$.

Let C_2 be a projective incidence structure, let a, b be points of C_2, and let M be a line of C_2. We say that a, b lie on M if and only if:

[3] The proposition (7) has been removed.

[4] The definition (Def. 4) has been removed.

(Def. 7) a lies on M and b lies on M.

Let c be a point of C_2. We say that a, b, c lie on M if and only if:

(Def. 8) a lies on M and b lies on M and c lies on M.

We now state three propositions:

(21) Suppose that for all points p_1, r_2, q, r_1, q_1, p, r of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear. Let p, q, r, s, a, b, c be points of Inc-ProjSp(C_1) and L, Q, R, S, A, B, C be lines of Inc-ProjSp(C_1). Suppose that q does not lie on L and r does not lie on L and p does not lie on Q and s does not lie on Q and p does not lie on R and r does not lie on R and q does not lie on S and s does not lie on S and a, p, s lie on L and a, q, r lie on Q and b, q, s lie on R and b, p, r lie on S and c, p, q lie on A and c, r, s lie on B and a, b lie on C. Then c does not lie on C.

(22) Suppose that for all points o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear. Let o, b_1, a_1, b_2, a_2, b_3, a_3, r, s, t be points of Inc-ProjSp(C_1) and C_3, C_4, C_5, A_1, A_2, A_3, B_1, B_2, B_3 be lines of Inc-ProjSp(C_1). Suppose that o, b_1, a_1 lie on C_3 and o, a_2, b_2 lie on C_4 and o, a_3, b_3 lie on C_5 and a_3, a_2, t lie on A_1 and a_3, r, a_1 lie on A_2 and a_2, s, a_1 lie on A_3 and t, b_2, b_3 lie on B_1 and b_1, r, b_3 lie on B_2 and b_1, s, b_2 lie on B_3 and C_3, C_4, C_5 are mutually different and $o \neq a_1$ and $o \neq a_2$ and $o \neq a_3$ and $o \neq b_1$ and $o \neq b_2$ and $o \neq b_3$ and $a_1 \neq b_1$ and $a_2 \neq b_2$ and $a_3 \neq b_3$. Then there exists a line O of Inc-ProjSp(C_1) such that r, s, t lie on O.

(23) Suppose that for all points o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear. Let o, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be points of Inc-ProjSp(C_1) and A_1, A_2, A_3, B_1, B_2, B_3, C_3, C_4, C_5 be lines of Inc-ProjSp(C_1). Suppose that o, a_1, a_2, a_3 are mutually different and o, b_1, b_2, b_3 are mutually different and $A_3 \neq B_3$ and o lies on A_3 and o lies on B_3 and a_2, b_3, c_1 lie on A_1 and a_3, b_1, c_2 lie on B_1 and a_1, b_2, c_3 lie on C_3 and a_1, b_3, c_2 lie on A_2 and a_3, b_2, c_1 lie on B_2 and a_2, b_1, c_3 lie on C_4 and b_1, b_2, b_3 lie on A_3 and a_1, a_2, a_3 lie on B_3 and c_1, c_2 lie on C_5. Then c_3 lies on C_5.

Let I_1 be a projective incidence structure. We say that I_1 is partial if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let p, q be points of I_1 and P, Q be lines of I_1. Suppose p lies on P and q lies on P and p lies on Q and q lies on Q. Then $p = q$ or $P = Q$.

We say that I_1 is linear if and only if:

(Def. 10) For all points p, q of I_1 there exists a line P of I_1 such that p lies on P and q lies on P.

We say that I_1 is at least 2-dimensional if and only if:

(Def. 11) There exists a point p of I_1 and there exists a line P of I_1 such that p does not lie on P.

We say that I_1 is at least 3-rank if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let P be a line of I_1. Then there exist points a, b, c of I_1 such that $a \neq b$ and $b \neq c$ and $c \neq a$ and a lies on P and b lies on P and c lies on P.

We say that I_1 is Vebleian if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let a, b, c, d, p, q be points of I_1 and M, N, P, Q be lines of I_1. Suppose that a lies on M and b lies on M and c lies on N and d lies on N and p lies on M and p lies on N and a lies on P and c lies on P and b lies on Q and d lies on Q and p does not lie on P and p does not lie on Q and $M \neq N$. Then there exists a point q of I_1 such that q lies on P and q lies on Q.

Let C_1 be a projective space defined in terms of collinearity. Note that Inc-ProjSp(C_1) is partial, linear, at least 2-dimensional, at least 3-rank, and Vebleian.

One can verify that there exists a projective incidence structure which is strict, partial, linear, at least 2-dimensional, at least 3-rank, and Vebleian.

A projective space defined in terms of incidence is a partial linear at least 2-dimensional at least 3-rank Vebleian projective incidence structure.

Let C_1 be a projective space defined in terms of collinearity. Observe that Inc-ProjSp(C_1) is partial, linear, at least 2-dimensional, at least 3-rank, and Vebleian.

Let I_1 be a projective space defined in terms of incidence. We say that I_1 is 2-dimensional if and only if:

(Def. 14) For all lines M, N of I_1 there exists a point q of I_1 such that q lies on M and q lies on N.

We introduce I_1 is up 3-dimensional as an antonym of I_1 is 2-dimensional.

Let I_1 be a projective space defined in terms of incidence. We say that I_1 is at most 3 dimensional if and only if the condition (Def. 16) is satisfied.

(Def. 16)[5] Let a be a point of I_1 and M, N be lines of I_1. Then there exist points b, c of I_1 and there exists a line S of I_1 such that a lies on S and b lies on S and c lies on S and b lies on M and c lies on N.

Let I_1 be a projective space defined in terms of incidence. We say that I_1 is 3-dimensional if and only if:

(Def. 17) I_1 is at most 3 dimensional and up 3-dimensional.

Let I_1 be a projective space defined in terms of incidence. We say that I_1 is Fanoian if and only if the condition (Def. 18) is satisfied.

(Def. 18) Let p, q, r, s, a, b, c be points of I_1 and L, Q, R, S, A, B, C be lines of I_1. Suppose that q does not lie on L and r does not lie on L and p does not lie on Q and s does not lie on Q and p does not lie on R and r does not lie on R and q does not lie on S and s does not lie on S and a, p, s lie on L and a, q, r lie on Q and b, q, s lie on R and b, p, r lie on S and c, p, q lie on A and c, r, s lie on B and a, b lie on C. Then c does not lie on C.

Let I_1 be a projective space defined in terms of incidence. We say that I_1 is Desarguesian if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let o, b_1, a_1, b_2, a_2, b_3, a_3, r, s, t be points of I_1 and C_3, C_4, C_5, A_1, A_2, A_3, B_1, B_2, B_3 be lines of I_1. Suppose that o, b_1, a_1 lie on C_3 and o, a_2, b_2 lie on C_4 and o, a_3, b_3 lie on C_5 and a_3, a_2, t lie on A_1 and a_3, r, a_1 lie on A_2 and a_2, s, a_1 lie on A_3 and t, b_2, b_3 lie on B_1 and b_1, r, b_3 lie on B_2 and b_1, s, b_2 lie on B_3 and C_3, C_4, C_5 are mutually different and $o \neq a_1$ and $o \neq a_2$ and $o \neq a_3$ and $o \neq b_1$ and $o \neq b_2$ and $o \neq b_3$ and $a_1 \neq b_1$ and $a_2 \neq b_2$ and $a_3 \neq b_3$. Then there exists a line O of I_1 such that r, s, t lie on O.

Let I_1 be a projective space defined in terms of incidence. We say that I_1 is Pappian if and only if the condition (Def. 20) is satisfied.

[5] The definition (Def. 15) has been removed.

(Def. 20) Let o, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be points of I_1 and A_1, A_2, A_3, B_1, B_2, B_3, C_3, C_4, C_5 be lines of I_1. Suppose that o, a_1, a_2, a_3 are mutually different and o, b_1, b_2, b_3 are mutually different and $A_3 \neq B_3$ and o lies on A_3 and o lies on B_3 and a_2, b_3, c_1 lie on A_1 and a_3, b_1, c_2 lie on B_1 and a_1, b_2, c_3 lie on C_3 and a_1, b_3, c_2 lie on A_2 and a_3, b_2, c_1 lie on B_2 and a_2, b_1, c_3 lie on C_4 and b_1, b_2, b_3 lie on A_3 and a_1, a_2, a_3 lie on B_3 and c_1, c_2 lie on C_5. Then c_3 lies on C_5.

One can check that there exists a projective space defined in terms of incidence which is Desarguesian, Fanoian, at most 3 dimensional, and up 3-dimensional.

One can check that there exists a projective space defined in terms of incidence which is Pappian, Fanoian, at most 3 dimensional, and up 3-dimensional.

One can verify that there exists a projective space defined in terms of incidence which is Desarguesian, Fanoian, and 2-dimensional.

Let us observe that there exists a projective space defined in terms of incidence which is Pappian, Fanoian, and 2-dimensional.

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