

On the Hausdorff Distance Between Compact Subsets¹

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Summary. In [2] the pseudo-metric $\text{dist}_{\min}^{\max}$ on compact subsets A and B of a topological space generated from arbitrary metric space is defined. Using this notion we define the Hausdorff distance (see e.g. [6]) of A and B as a maximum of the two pseudo-distances: from A to B and from B to A . We justify its distance properties. At the end we define some special notions which enable to apply the Hausdorff distance operator “HausDist” to the subsets of the Euclidean topological space \mathcal{E}_T^n .

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The articles [16], [19], [1], [17], [11], [18], [20], [4], [15], [7], [10], [9], [12], [3], [8], [5], [2], [14], and [13] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let r be a real number. Then $\{r\}$ is a subset of \mathbb{R} .

Let M be a non empty metric space. Note that M_{top} is T_2 .

Next we state a number of propositions:

- (1) For all real numbers x, y such that $x \geq 0$ and $y \geq 0$ and $\max(x, y) = 0$ holds $x = 0$.
- (2) For every non empty metric space M and for every point x of M holds $(\text{dist}(x))(x) = 0$.
- (3) For every non empty metric space M and for every subset P of M_{top} and for every point x of M such that $x \in P$ holds $0 \in (\text{dist}(x))^{\circ}P$.
- (4) Let M be a non empty metric space, P be a subset of M_{top} , x be a point of M , and y be a real number. If $y \in (\text{dist}(x))^{\circ}P$, then $y \geq 0$.
- (5) For every non empty metric space M and for every subset P of M_{top} and for every set x such that $x \in P$ holds $(\text{dist}_{\min}(P))(x) = 0$.
- (6) Let M be a non empty metric space, p be a point of M , q be a point of M_{top} , and r be a real number. If $p = q$ and $r > 0$, then $\text{Ball}(p, r)$ is a neighbourhood of q .
- (7) Let M be a non empty metric space, A be a subset of M_{top} , and p be a point of M . Then $p \in \bar{A}$ if and only if for every real number r such that $r > 0$ holds $\text{Ball}(p, r)$ meets A .

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- (8) Let M be a non empty metric space, p be a point of M , and A be a subset of M_{top} . Then $p \in \bar{A}$ if and only if for every real number r such that $r > 0$ there exists a point q of M such that $q \in A$ and $\rho(p, q) < r$.
- (9) Let M be a non empty metric space, P be a non empty subset of M_{top} , and x be a point of M . Then $(\text{dist}_{\min}(P))(x) = 0$ if and only if for every real number r such that $r > 0$ there exists a point p of M such that $p \in P$ and $\rho(x, p) < r$.
- (10) Let M be a non empty metric space, P be a non empty subset of M_{top} , and x be a point of M . Then $x \in \bar{P}$ if and only if $(\text{dist}_{\min}(P))(x) = 0$.
- (11) Let M be a non empty metric space, P be a non empty closed subset of M_{top} , and x be a point of M . Then $x \in P$ if and only if $(\text{dist}_{\min}(P))(x) = 0$.
- (12) For every non empty subset A of \mathbb{R}^1 there exists a non empty subset X of \mathbb{R} such that $A = X$ and $\inf A = \inf X$.
- (13) For every non empty subset A of \mathbb{R}^1 there exists a non empty subset X of \mathbb{R} such that $A = X$ and $\sup A = \sup X$.
- (14) Let M be a non empty metric space, P be a non empty subset of M_{top} , x be a point of M , and X be a subset of \mathbb{R} . If $X = (\text{dist}(x))^{\circ} P$, then X is lower bounded.
- (15) Let M be a non empty metric space, P be a non empty subset of M_{top} , and x, y be points of M . If $y \in P$, then $(\text{dist}_{\min}(P))(x) \leq \rho(x, y)$.
- (16) Let M be a non empty metric space, P be a non empty subset of M_{top} , r be a real number, and x be a point of M . If for every point y of M such that $y \in P$ holds $\rho(x, y) \geq r$, then $(\text{dist}_{\min}(P))(x) \geq r$.
- (17) Let M be a non empty metric space, P be a non empty subset of M_{top} , and x, y be points of M . Then $(\text{dist}_{\min}(P))(x) \leq \rho(x, y) + (\text{dist}_{\min}(P))(y)$.
- (18) Let M be a non empty metric space, P be a subset of M_{top} , and Q be a non empty subset of M . If $P = Q$, then $M_{\text{top}} \upharpoonright P = (M \upharpoonright Q)_{\text{top}}$.
- (19) Let M be a non empty metric space, A be a subset of M , B be a non empty subset of M , and C be a subset of $M \upharpoonright B$. If $A \subseteq B$ and $A = C$ and C is bounded, then A is bounded.
- (20) Let M be a non empty metric space, B be a subset of M , and A be a subset of M_{top} . If $A = B$ and A is compact, then B is bounded.
- (21) Let M be a non empty metric space, P be a non empty subset of M_{top} , and z be a point of M . Then there exists a point w of M such that $w \in P$ and $(\text{dist}_{\min}(P))(z) \leq \rho(w, z)$.

Let M be a non empty metric space and let x be a point of M . Note that $\text{dist}(x)$ is continuous.

Let M be a non empty metric space and let X be a compact non empty subset of M_{top} . One can verify that $\text{dist}_{\max}(X)$ is continuous and $\text{dist}_{\min}(X)$ is continuous.

One can prove the following propositions:

- (22) Let M be a non empty metric space, P be a non empty subset of M_{top} , and x, y be points of M . If $y \in P$ and P is compact, then $(\text{dist}_{\max}(P))(x) \geq \rho(x, y)$.
- (23) Let M be a non empty metric space, P be a non empty subset of M_{top} , and z be a point of M . If P is compact, then there exists a point w of M such that $w \in P$ and $(\text{dist}_{\max}(P))(z) \geq \rho(w, z)$.
- (24) Let M be a non empty metric space, P, Q be non empty subsets of M_{top} , and z be a point of M . If P is compact and Q is compact and $z \in Q$, then $(\text{dist}_{\min}(P))(z) \leq \text{dist}_{\max}^{\text{max}}(P, Q)$.
- (25) Let M be a non empty metric space, P, Q be non empty subsets of M_{top} , and z be a point of M . If P is compact and Q is compact and $z \in Q$, then $(\text{dist}_{\max}(P))(z) \leq \text{dist}_{\max}^{\text{max}}(P, Q)$.

- (26) Let M be a non empty metric space, P, Q be non empty subsets of M_{top} , and X be a subset of \mathbb{R} . If $X = (\text{dist}_{\max}(P))^{\circ}Q$ and P is compact and Q is compact, then X is upper bounded.
- (27) Let M be a non empty metric space, P, Q be non empty subsets of M_{top} , and X be a subset of \mathbb{R} . If $X = (\text{dist}_{\min}(P))^{\circ}Q$ and P is compact and Q is compact, then X is upper bounded.
- (28) Let M be a non empty metric space, P be a non empty subset of M_{top} , and z be a point of M . If P is compact, then $(\text{dist}_{\min}(P))(z) \leq (\text{dist}_{\max}(P))(z)$.
- (29) For every non empty metric space M and for every non empty subset P of M_{top} holds $(\text{dist}_{\min}(P))^{\circ}P = \{0\}$.
- (30) Let M be a non empty metric space and P, Q be non empty subsets of M_{top} . If P is compact and Q is compact, then $\text{dist}_{\min}^{\max}(P, Q) \geq 0$.
- (31) For every non empty metric space M and for every non empty subset P of M_{top} holds $\text{dist}_{\min}^{\max}(P, P) = 0$.
- (32) Let M be a non empty metric space and P, Q be non empty subsets of M_{top} . If P is compact and Q is compact, then $\text{dist}_{\max}^{\min}(P, Q) \geq 0$.
- (33) Let M be a non empty metric space, Q, R be non empty subsets of M_{top} , and y be a point of M . If Q is compact and R is compact and $y \in Q$, then $(\text{dist}_{\min}(R))(y) \leq \text{dist}_{\min}^{\max}(R, Q)$.

2. THE HAUSDORFF DISTANCE

Let M be a non empty metric space and let P, Q be subsets of M_{top} . The functor $\text{HausDist}(P, Q)$ yields a real number and is defined by:

(Def. 1) $\text{HausDist}(P, Q) = \max(\text{dist}_{\min}^{\max}(P, Q), \text{dist}_{\min}^{\max}(Q, P))$.

Let us note that the functor $\text{HausDist}(P, Q)$ is commutative.

One can prove the following propositions:

- (34) Let M be a non empty metric space, Q, R be non empty subsets of M_{top} , and y be a point of M . If Q is compact and R is compact and $y \in Q$, then $(\text{dist}_{\min}(R))(y) \leq \text{HausDist}(Q, R)$.
- (35) Let M be a non empty metric space and P, Q, R be non empty subsets of M_{top} . If P is compact and Q is compact and R is compact, then $\text{dist}_{\min}^{\max}(P, R) \leq \text{HausDist}(P, Q) + \text{HausDist}(Q, R)$.
- (36) Let M be a non empty metric space and P, Q, R be non empty subsets of M_{top} . If P is compact and Q is compact and R is compact, then $\text{dist}_{\min}^{\max}(R, P) \leq \text{HausDist}(P, Q) + \text{HausDist}(Q, R)$.
- (37) Let M be a non empty metric space and P, Q be non empty subsets of M_{top} . If P is compact and Q is compact, then $\text{HausDist}(P, Q) \geq 0$.
- (38) For every non empty metric space M and for every non empty subset P of M_{top} holds $\text{HausDist}(P, P) = 0$.
- (39) Let M be a non empty metric space and P, Q be non empty subsets of M_{top} . If P is compact and Q is compact and $\text{HausDist}(P, Q) = 0$, then $P = Q$.
- (40) Let M be a non empty metric space and P, Q, R be non empty subsets of M_{top} . If P is compact and Q is compact and R is compact, then $\text{HausDist}(P, R) \leq \text{HausDist}(P, Q) + \text{HausDist}(Q, R)$.

Let n be a natural number and let P, Q be subsets of \mathcal{E}_T^n . The functor $\text{dist}_{\min}^{\max}(P, Q)$ yields a real number and is defined as follows:

(Def. 2) There exist subsets P', Q' of $(\mathcal{E}^n)_{\text{top}}$ such that $P = P'$ and $Q = Q'$ and $\text{dist}_{\min}^{\max}(P, Q) = \text{dist}_{\min}^{\max}(P', Q')$.

Let n be a natural number and let P, Q be subsets of \mathcal{E}_T^n . The functor $\text{HausDist}(P, Q)$ yields a real number and is defined by:

(Def. 3) There exist subsets P', Q' of $(\mathcal{E}^n)_{\text{top}}$ such that $P = P'$ and $Q = Q'$ and $\text{HausDist}(P, Q) = \text{HausDist}(P', Q')$.

Let us notice that the functor $\text{HausDist}(P, Q)$ is commutative.

In the sequel n is a natural number.

The following propositions are true:

- (41) For all non empty subsets P, Q of \mathcal{E}_T^n such that P is compact and Q is compact holds $\text{HausDist}(P, Q) \geq 0$.
- (42) For every non empty subset P of \mathcal{E}_T^n holds $\text{HausDist}(P, P) = 0$.
- (43) For all non empty subsets P, Q of \mathcal{E}_T^n such that P is compact and Q is compact and $\text{HausDist}(P, Q) = 0$ holds $P = Q$.
- (44) For all non empty subsets P, Q, R of \mathcal{E}_T^n such that P is compact and Q is compact and R is compact holds $\text{HausDist}(P, R) \leq \text{HausDist}(P, Q) + \text{HausDist}(Q, R)$.

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