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Inst. of Computer Science, Univ. of Białystok

Hahn-Banach Theorem

Bogdan Nowak
Łódź University

Andrzej Trybulec
Warsaw University
Białystok

Summary. We prove a version of Hahn-Banach Theorem.

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The articles [13], [6], [19], [1], [7], [9], [20], [3], [4], [17], [16], [15], [10], [5], [11], [8], [18], [14], [12], and [2] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(2)[1] For every set X and for all functions f, g such that $X \subseteq \operatorname{dom} f$ and $f \subseteq g$ holds $f \upharpoonright X = g \upharpoonright X$.

(3) For every non empty set A and for every set b such that $A \neq \{b\}$ there exists an element a of A such that $a \neq b$.

(4) For all sets X, Y holds every non empty subset of $X \dot{\rightarrow} Y$ is a non empty functional set.

(5) Let B be a non empty functional set and f be a function. Suppose $f = \bigcup B$. Then $\operatorname{dom} f = \bigcup\{\operatorname{dom} g : g$ ranges over elements of $B\}$ and $\operatorname{rng} f = \bigcup\{\operatorname{rng} g : g$ ranges over elements of $B\}$.

(6) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every extended real number r such that $r \in A$ holds $r \leq -\infty$ holds $A = \{-\infty\}$.

(7) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every extended real number r such that $r \in A$ holds $+\infty \leq r$ holds $A = \{+\infty\}$.

(8) Let A be a non empty subset of $\overline{\mathbb{R}}$ and r be an extended real number. If $r < \sup A$, then there exists an extended real number s such that $s \in A$ and $r < s$.

(9) Let A be a non empty subset of $\overline{\mathbb{R}}$ and r be an extended real number. If $\inf A < r$, then there exists an extended real number s such that $s \in A$ and $s < r$.

(10) Let A, B be non empty subsets of $\overline{\mathbb{R}}$. Suppose that for all extended real numbers r, s such that $r \in A$ and $s \in B$ holds $r \leq s$. Then $\sup A \leq \inf B$.

(12)[2] Let x, y be extended real numbers and p, q be real numbers. If $x = p$ and $y = q$, then $p \leq q$ iff $x \leq y$.

[1] The proposition (1) has been removed.
[2] The proposition (11) has been removed.

2. Sets Linearly Ordered by the Inclusion

Let A be a non empty set. Note that there exists a subset of A which is $\subseteq$-linear and non empty.

One can prove the following proposition

(13) For all sets X, Y and for every $\subseteq$-linear subset B of $X \dot{\to} Y$ holds $\bigcup B \in X \dot{\to} Y$.

3. Subspaces of a Real Linear Space

In the sequel V denotes a real linear space.

Next we state a number of propositions:

(14) For all subspaces W_1, W_2 of V holds the carrier of $W_1 \subseteq$ the carrier of $W_1 + W_2$.

(15) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(16) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(17) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(18) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_{\langle W_2, W_1 \rangle} = \langle v_2, v_1 \rangle$.

(19) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v_{\langle W_1, W_2 \rangle} = \langle v, 0_V \rangle$.

(20) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v_{\langle W_1, W_2 \rangle} = \langle 0_V, v \rangle$.

(21) Let V_1 be a subspace of V, W_1 be a subspace of V_1, and v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(22) For all subspaces V_1, V_2, W of V and for all subspaces W_1, W_2 of W such that $W_1 = V_1$ and $W_2 = V_2$ holds $W_1 + W_2 = V_1 + V_2$.

(23) For every subspace W of V and for every vector v of V and for every vector w of W such that $v = w$ holds $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(24) Let v be a vector of V and X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(25) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle 0_W, y \rangle$.

(26) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle w, 0_V \rangle$.

(27) For every vector v of V and for all subspaces W_1, W_2 of V there exist vectors v_1, v_2 of V such that $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(28) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists a real number r such that $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$.

(29) Let v be a vector of V, X be a subspace of V, y be a vector of $X+\mathrm{Lin}(\{v\})$, and W be a subspace of $X+\mathrm{Lin}(\{v\})$. Suppose $v=y$ and $X=W$ and $v\notin X$. Let w_1, w_2 be vectors of $X+\mathrm{Lin}(\{v\})$, x_1, x_2 be vectors of X, and r_1, r_2 be real numbers. If $(w_1)_{\langle W,\mathrm{Lin}(\{y\})\rangle}=\langle x_1, r_1\cdot v\rangle$ and $(w_2)_{\langle W,\mathrm{Lin}(\{y\})\rangle}=\langle x_2, r_2\cdot v\rangle$, then $(w_1+w_2)_{\langle W,\mathrm{Lin}(\{y\})\rangle}=\langle x_1+x_2, (r_1+r_2)\cdot v\rangle$.

(30) Let v be a vector of V, X be a subspace of V, y be a vector of $X+\mathrm{Lin}(\{v\})$, and W be a subspace of $X+\mathrm{Lin}(\{v\})$. Suppose $v=y$ and $X=W$ and $v\notin X$. Let w be a vector of $X+\mathrm{Lin}(\{v\})$, x be a vector of X, and t, r be real numbers. If $w_{\langle W,\mathrm{Lin}(\{y\})\rangle}=\langle x, r\cdot v\rangle$, then $(t\cdot w)_{\langle W,\mathrm{Lin}(\{y\})\rangle}=\langle t\cdot x, t\cdot r\cdot v\rangle$.

4. FUNCTIONALS

Let V be an RLS structure. A functional in V is a function from the carrier of V into $\mathbb{R}$.

Let us consider V and let I_1 be a functional in V. We say that I_1 is subadditive if and only if:

(Def. 3)[3] For all vectors x, y of V holds $I_1(x+y)\leq I_1(x)+I_1(y)$.

We say that I_1 is additive if and only if:

(Def. 4) For all vectors x, y of V holds $I_1(x+y)=I_1(x)+I_1(y)$.

We say that I_1 is homogeneous if and only if:

(Def. 5) For every vector x of V and for every real number r holds $I_1(r\cdot x)=r\cdot I_1(x)$.

We say that I_1 is positively homogeneous if and only if:

(Def. 6) For every vector x of V and for every real number r such that $r>0$ holds $I_1(r\cdot x)=r\cdot I_1(x)$.

We say that I_1 is semi-homogeneous if and only if:

(Def. 7) For every vector x of V and for every real number r such that $r\geq 0$ holds $I_1(r\cdot x)=r\cdot I_1(x)$.

We say that I_1 is absolutely homogeneous if and only if:

(Def. 8) For every vector x of V and for every real number r holds $I_1(r\cdot x)=|r|\cdot I_1(x)$.

We say that I_1 is 0-preserving if and only if:

(Def. 9) $I_1(0_V)=0$.

Let us consider V. One can check the following observations:

* every functional in V which is additive is also subadditive,
* every functional in V which is homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also 0-preserving,
* every functional in V which is absolutely homogeneous is also semi-homogeneous, and
* every functional in V which is 0-preserving and positively homogeneous is also semi-homogeneous.

Let us consider V. Note that there exists a functional in V which is additive, absolutely homogeneous, and homogeneous.

Let us consider V. A Banach functional in V is a subadditive positively homogeneous functional in V. A linear functional in V is an additive homogeneous functional in V.

One can prove the following four propositions:

[3] The definitions (Def. 1) and (Def. 2) have been removed.

(31) For every homogeneous functional L in V and for every vector v of V holds $L(-v) = -L(v)$.

(32) For every linear functional L in V and for all vectors v_1, v_2 of V holds $L(v_1 - v_2) = L(v_1) - L(v_2)$.

(33) For every additive functional L in V holds $L(0_V) = 0$.

(34) Let X be a subspace of V, f_1 be a linear functional in X, v be a vector of V, and y be a vector of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be a real number. Then there exists a linear functional p_1 in $X + \mathrm{Lin}(\{v\})$ such that $p_1 \restriction$the carrier of $X = f_1$ and $p_1(y) = r$.

5. Hahn-Banach Theorem

We now state three propositions:

(35) Let V be a real linear space, X be a subspace of V, q be a Banach functional in V, and f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq q(v)$. Then there exists a linear functional p_1 in V such that $p_1 \restriction$the carrier of $X = f_1$ and for every vector x of V holds $p_1(x) \leq q(x)$.

(36) For every real normed space V holds the norm of V is an absolutely homogeneous subadditive functional in V.

(37) Let V be a real normed space, X be a subspace of V, and f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq \|v\|$. Then there exists a linear functional p_1 in V such that $p_1 \restriction$the carrier of $X = f_1$ and for every vector x of V holds $p_1(x) \leq \|x\|$.

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