Solvable Groups

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Summary.   The concept of solvable group is introduced. Some theorems concerning heirdom of solvability are proved.

The articles [6], [12], [13], [2], [3], [7], [5], [4], [1], [9], [10], [8], and [11] provide the notation and terminology for this paper.

In this paper i denotes a natural number.

Let I_1 be a group. We say that I_1 is solvable if and only if the condition (Def. 1) is satisfied.

(Def. 1)   There exists a finite sequence F of elements of $\operatorname{SubGr} I_1$ such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_{(I_1)}$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_{(I_1)}$, and

(iv)   for every i such that $i \in \operatorname{dom} F$ and $i+1 \in \operatorname{dom} F$ and for all strict subgroups G_1, G_2 of I_1 such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds $^{G_1}/_N$ is commutative.

One can verify that there exists a group which is solvable and strict.

The following propositions are true:

(1)   Let G be a strict group and H, F_1, F_2 be strict subgroups of G. Suppose F_1 is a normal subgroup of F_2. Then $F_1 \cap H$ is a normal subgroup of $F_2 \cap H$.

(2)   Let G be a strict group, F_2 be a strict subgroup of G, F_1 be a strict normal subgroup of F_2, and a, b be elements of F_2. Then $a \cdot F_1 \cdot (b \cdot F_1) = (a \cdot b) \cdot F_1$.

(3)   Let G be a strict group, H, F_2 be strict subgroups of G, F_1 be a strict normal subgroup of F_2, and G_2 be a strict subgroup of G. Suppose $G_2 = H \cap F_2$. Let G_1 be a normal subgroup of G_2. Suppose $G_1 = H \cap F_1$. Then there exists a subgroup G_3 of $^{F_2}/_{F_1}$ such that $^{G_2}/_{G_1}$ and G_3 are isomorphic.

(4)   Let G be a strict group, H, F_2 be strict subgroups of G, F_1 be a strict normal subgroup of F_2, and G_2 be a strict subgroup of G. Suppose $G_2 = F_2 \cap H$. Let G_1 be a normal subgroup of G_2. Suppose $G_1 = F_1 \cap H$. Then there exists a subgroup G_3 of $^{F_2}/_{F_1}$ such that $^{G_2}/_{G_1}$ and G_3 are isomorphic.

(5)   For every solvable strict group G holds every strict subgroup of G is solvable.

(6) Let G be a strict group. Given a finite sequence F of elements of $\operatorname{SubGr} G$ such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_G$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and

(iv) for every i such that $i \in \operatorname{dom} F$ and $i + 1 \in \operatorname{dom} F$ and for all strict subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a strict normal subgroup of G_1 and for every normal subgroup N of G_1 such that $N = G_2$ holds $^{G_1}/_N$ is a cyclic group.

Then G is solvable.

(7) Every strict commutative group is strict and solvable.

Let G, H be strict groups, let g be a homomorphism from G to H, and let A be a subgroup of G. The functor $g{\restriction}A$ yielding a homomorphism from A to H is defined as follows:

(Def. 2) $g{\restriction}A = g{\restriction}$the carrier of A.

Let G, H be strict groups, let g be a homomorphism from G to H, and let A be a subgroup of G. The functor $g^\circ A$ yields a strict subgroup of H and is defined as follows:

(Def. 3) $g^\circ A = \operatorname{Im}(g{\restriction}A)$.

The following propositions are true:

(8) Let G, H be strict groups, g be a homomorphism from G to H, and A be a subgroup of G. Then $\operatorname{rng}(g{\restriction}A) = g^\circ$(the carrier of A).

(9) Let G, H be strict groups, g be a homomorphism from G to H, and A be a strict subgroup of G. Then the carrier of $g^\circ A = g^\circ$(the carrier of A).

(10) Let G, H be strict groups, h be a homomorphism from G to H, and A be a strict subgroup of G. Then $\operatorname{Im}(h{\restriction}A)$ is a strict subgroup of $\operatorname{Im} h$.

(11) Let G, H be strict groups, h be a homomorphism from G to H, and A be a strict subgroup of G. Then $h^\circ A$ is a strict subgroup of $\operatorname{Im} h$.

(12) For all strict groups G, H and for every homomorphism h from G to H holds $h^\circ(\{\mathbf{1}\}_G) = \{\mathbf{1}\}_H$ and $h^\circ(\Omega_G) = \Omega_{\operatorname{Im} h}$.

(13) Let G, H be strict groups, h be a homomorphism from G to H, and A, B be strict subgroups of G. If A is a subgroup of B, then $h^\circ A$ is a subgroup of $h^\circ B$.

(14) Let G, H be strict groups, h be a homomorphism from G to H, A be a strict subgroup of G, and a be an element of G. Then $h(a) \cdot h^\circ A = h^\circ(a \cdot A)$ and $h^\circ A \cdot h(a) = h^\circ(A \cdot a)$.

(15) Let G, H be strict groups, h be a homomorphism from G to H, and A, B be subsets of G. Then $h^\circ A \cdot h^\circ B = h^\circ(A \cdot B)$.

(16) Let G, H be strict groups, h be a homomorphism from G to H, and A, B be strict subgroups of G. Suppose A is a strict normal subgroup of B. Then $h^\circ A$ is a strict normal subgroup of $h^\circ B$.

(17) Let G, H be strict groups and h be a homomorphism from G to H. If G is a solvable group, then $\operatorname{Im} h$ is solvable.

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