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Homomorphisms and Isomorphisms of Groups. Quotient Group

Wojciech A. Trybulec
Warsaw University

Michał J. Trybulec
Warsaw University

Summary. Quotient group, homomorphisms and isomorphisms of groups are introduced. The so called isomorphism theorems are proved following [9].

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The articles [11], [7], [19], [20], [4], [6], [5], [1], [8], [3], [2], [12], [13], [10], [15], [17], [14], [16], and [18] provide the notation and terminology for this paper.

One can prove the following proposition

(1) Let A, B be non empty sets and f be a function from A into B. Then f is one-to-one if and only if for all elements a, b of A such that $f(a) = f(b)$ holds $a = b$.

Let G be a group and let A be a subgroup of G. We see that the subgroup of A is a subgroup of G.

Let G be a group. One can verify that $\{\mathbf{1}\}_G$ is normal and Ω_G is normal.

For simplicity, we follow the rules: n denotes a natural number, i denotes an integer, G, H, I denote groups, A, B denote subgroups of G, N denotes a normal subgroup of G, a, a_1, a_2, a_3, b denote elements of G, c denotes an element of H, x denotes a set, and A_1, A_2 denote subsets of G.

We now state several propositions:

(2) Let X be a subgroup of A and x be an element of A. Suppose $x = a$. Then $x \cdot X = a \cdot (X$ **qua** subgroup of $G)$ and $X \cdot x = (X$ **qua** subgroup of $G) \cdot a$.

(3) For all subgroups X, Y of A holds $(X$ **qua** subgroup of $G) \cap (Y$ **qua** subgroup of $G) = X \cap Y$.

(4) $a \cdot b \cdot a^{-1} = b^{a^{-1}}$ and $a \cdot (b \cdot a^{-1}) = b^{a^{-1}}$.

(6)[1] $a \cdot A \cdot A = a \cdot A$ and $a \cdot (A \cdot A) = a \cdot A$ and $A \cdot A \cdot a = A \cdot a$ and $A \cdot (A \cdot a) = A \cdot a$.

(7) Let G be a group and A_1 be a subset of G. If $A_1 = \{[a,b] : a$ ranges over elements of G, b ranges over elements of $G\}$, then $G^{\rm c} = \mathrm{gr}(A_1)$.

(8) Let G be a strict group and B be a strict subgroup of G. Then $G^{\rm c}$ is a subgroup of B if and only if for all elements a, b of G holds $[a,b] \in B$.

(9) For every normal subgroup N of G such that N is a subgroup of B holds N is a normal subgroup of B.

[1] The proposition (5) has been removed.

Let us consider G, B and let M be a normal subgroup of G. Let us assume that the groupoid of M is a subgroup of B. The functor $(M)_B$ yields a strict normal subgroup of B and is defined by:

(Def. 1) $(M)_B =$ the groupoid of M.

We now state the proposition

(10) $B \cap N$ is a normal subgroup of B and $N \cap B$ is a normal subgroup of B.

Let us consider G, B and let N be a normal subgroup of G. Then $B \cap N$ is a strict normal subgroup of B.

Let us consider G, let N be a normal subgroup of G, and let us consider B. Then $N \cap B$ is a strict normal subgroup of B.

Let G be a non empty 1-sorted structure. Let us observe that G is trivial if and only if:

(Def. 2) There exists x such that the carrier of $G = \{x\}$.

Let us observe that there exists a group which is trivial.

Next we state three propositions:

(11) $\{\mathbf{1}\}_G$ is trivial.

(12) G is trivial iff $\mathrm{ord}(G) = 1$ and G is finite.

(13) For every strict group G such that G is trivial holds $\{\mathbf{1}\}_G = G$.

Let us consider G, N. The functor $\mathrm{Cosets}\, N$ yielding a set is defined as follows:

(Def. 3) $\mathrm{Cosets}\, N =$ the left cosets of N.

Let us consider G, N. Observe that $\mathrm{Cosets}\, N$ is non empty.

We now state several propositions:

(14) For every normal subgroup N of G holds $\mathrm{Cosets}\, N =$ the left cosets of N and $\mathrm{Cosets}\, N =$ the right cosets of N.

(15) For every normal subgroup N of G such that $x \in \mathrm{Cosets}\, N$ there exists a such that $x = a \cdot N$ and $x = N \cdot a$.

(16) For every normal subgroup N of G holds $a \cdot N \in \mathrm{Cosets}\, N$ and $N \cdot a \in \mathrm{Cosets}\, N$.

(17) For every normal subgroup N of G such that $x \in \mathrm{Cosets}\, N$ holds x is a subset of G.

(18) For every normal subgroup N of G such that $A_1 \in \mathrm{Cosets}\, N$ and $A_2 \in \mathrm{Cosets}\, N$ holds $A_1 \cdot A_2 \in \mathrm{Cosets}\, N$.

Let us consider G and let N be a normal subgroup of G. The functor $\mathrm{CosOp}\, N$ yields a binary operation on $\mathrm{Cosets}\, N$ and is defined as follows:

(Def. 4) For all elements W_1, W_2 of $\mathrm{Cosets}\, N$ and for all A_1, A_2 such that $W_1 = A_1$ and $W_2 = A_2$ holds $(\mathrm{CosOp}\, N)(W_1, W_2) = A_1 \cdot A_2$.

Let us consider G and let N be a normal subgroup of G. The functor ${}^{G}/_{N}$ yields a groupoid and is defined by:

(Def. 5) ${}^{G}/_{N} = \langle \mathrm{Cosets}\, N, \mathrm{CosOp}\, N \rangle$.

Let us consider G and let N be a normal subgroup of G. Observe that ${}^{G}/_{N}$ is strict and non empty.

Next we state two propositions:

(22)[2] For every normal subgroup N of G holds the carrier of ${}^{G}/_{N} = \mathrm{Cosets}\, N$.

[2] The propositions (19)–(21) have been removed.

(23) For every normal subgroup N of G holds the multiplication of ${}^{G}/_{N} = \mathrm{CosOp}N$.

In the sequel N denotes a normal subgroup of G and S, T_1, T_2 denote elements of ${}^{G}/_{N}$.

Let us consider G, N, S. The functor ${}^{@}S$ yields a subset of G and is defined as follows:

(Def. 6) ${}^{@}S = S$.

The following two propositions are true:

(24) For every normal subgroup N of G and for all elements T_1, T_2 of ${}^{G}/_{N}$ holds $({}^{@}T_1) \cdot ({}^{@}T_2) = T_1 \cdot T_2$.

(25) ${}^{@}T_1 \cdot T_2 = ({}^{@}T_1) \cdot ({}^{@}T_2)$.

Let us consider G and let N be a normal subgroup of G. One can verify that ${}^{G}/_{N}$ is associative and group-like.

We now state a number of propositions:

(26) For every normal subgroup N of G and for every element S of ${}^{G}/_{N}$ there exists a such that $S = a \cdot N$ and $S = N \cdot a$.

(27) $N \cdot a$ is an element of ${}^{G}/_{N}$ and $a \cdot N$ is an element of ${}^{G}/_{N}$ and $\overline{N}$ is an element of ${}^{G}/_{N}$.

(28) For every normal subgroup N of G holds $x \in {}^{G}/_{N}$ iff there exists a such that $x = a \cdot N$ and $x = N \cdot a$.

(29) For every normal subgroup N of G holds $1_{{}^{G}/_{N}} = \overline{N}$.

(30) For every normal subgroup N of G and for every element S of ${}^{G}/_{N}$ such that $S = a \cdot N$ holds $S^{-1} = a^{-1} \cdot N$.

(31) For every normal subgroup N of G such that the left cosets of N is finite holds ${}^{G}/_{N}$ is finite.

(32) For every normal subgroup N of G holds $\mathrm{Ord}({}^{G}/_{N}) = |\bullet : N|$.

(33) For every normal subgroup N of G such that the left cosets of N is finite holds $\mathrm{ord}({}^{G}/_{N}) = |\bullet : N|_{\mathbb{N}}$.

(34) For every strict normal subgroup M of G such that M is a subgroup of B holds ${}^{B}/_{(M)_B}$ is a subgroup of ${}^{G}/_{M}$.

(35) Let N, M be strict normal subgroups of G. If M is a subgroup of N, then ${}^{N}/_{(M)_N}$ is a normal subgroup of ${}^{G}/_{M}$.

(36) Let G be a strict group and N be a strict normal subgroup of G. Then ${}^{G}/_{N}$ is a commutative group if and only if G^{c} is a subgroup of N.

Let us consider G, H. A function from the carrier of G into the carrier of H is said to be a homomorphism from G to H if:

(Def. 7) $\mathrm{It}(a \cdot b) = \mathrm{it}(a) \cdot \mathrm{it}(b)$.

In the sequel g, h are homomorphisms from G to H, g_1 is a homomorphism from H to G, and h_1 is a homomorphism from H to I.

We now state several propositions:

(40)[3] $g(1_G) = 1_H$.

(41) $g(a^{-1}) = g(a)^{-1}$.

(42) $g(a^b) = g(a)^{g(b)}$.

[3] The propositions (37)–(39) have been removed.

(43) $g([a,b]) = [g(a), g(b)]$.

(44) $g([a_1, a_2, a_3]) = [g(a_1), g(a_2), g(a_3)]$.

(45) $g(a^n) = g(a)^n$.

(46) $g(a^i) = g(a)^i$.

(47) $\mathrm{id}_{\text{the carrier of } G}$ is a homomorphism from G to G.

(48) $h_1 \cdot h$ is a homomorphism from G to I.

Let us consider G, H, I, h, h_1. Then $h_1 \cdot h$ is a homomorphism from G to I.
Let us consider G, H, g. Then $\mathrm{rng}\, g$ is a subset of H.
Let us consider G, H. The functor $G \to \{\mathbf{1}\}_H$ yields a homomorphism from G to H and is defined by:

(Def. 8) For every a holds $(G \to \{\mathbf{1}\}_H)(a) = 1_H$.

Next we state the proposition

(49) $h_1 \cdot (G \to \{\mathbf{1}\}_H) = G \to \{\mathbf{1}\}_I$ and $(H \to \{\mathbf{1}\}_I) \cdot h = G \to \{\mathbf{1}\}_I$.

Let us consider G and let N be a normal subgroup of G. The canonical homomorphism onto cosets of N yields a homomorphism from G to ${}^{G}/_{N}$ and is defined by:

(Def. 9) For every a holds (the canonical homomorphism onto cosets of $N)(a) = a \cdot N$.

Let us consider G, H, g. The functor $\mathrm{Ker}\, g$ yields a strict subgroup of G and is defined as follows:

(Def. 10) The carrier of $\mathrm{Ker}\, g = \{a : g(a) = 1_H\}$.

Let us consider G, H, g. Observe that $\mathrm{Ker}\, g$ is normal.
The following propositions are true:

(50) $a \in \mathrm{Ker}\, h$ iff $h(a) = 1_H$.

(51) For all strict groups G, H holds $\mathrm{Ker}(G \to \{\mathbf{1}\}_H) = G$.

(52) For every strict normal subgroup N of G holds Ker(the canonical homomorphism onto cosets of $N) = N$.

Let us consider G, H, g. The functor $\mathrm{Im}\, g$ yields a strict subgroup of H and is defined as follows:

(Def. 11) The carrier of $\mathrm{Im}\, g = g^{\circ}$(the carrier of G).

Next we state a number of propositions:

(53) $\mathrm{rng}\, g =$ the carrier of $\mathrm{Im}\, g$.

(54) $x \in \mathrm{Im}\, g$ iff there exists a such that $x = g(a)$.

(55) $\mathrm{Im}\, g = \mathrm{gr}(\mathrm{rng}\, g)$.

(56) $\mathrm{Im}(G \to \{\mathbf{1}\}_H) = \{\mathbf{1}\}_H$.

(57) For every normal subgroup N of G holds Im(the canonical homomorphism onto cosets of $N) = {}^{G}/_{N}$.

(58) h is a homomorphism from G to $\mathrm{Im}\, h$.

(59) If G is finite, then $\mathrm{Im}\, g$ is finite.

(60) If G is a commutative group, then $\mathrm{Im}\, g$ is commutative.

(61) $\mathrm{Ord}(\mathrm{Im}\, g) \leq \mathrm{Ord}(G)$.

(62) If G is finite, then $\mathrm{ord}(\mathrm{Im}\, g) \leq \mathrm{ord}(G)$.

Let us consider G, H, h. We say that h is monomorphism if and only if:

(Def. 12) h is one-to-one.

We introduce h is a monomorphism as a synonym of h is monomorphism. We say that h is epimorphism if and only if:

(Def. 13) $\mathrm{rng}\, h =$ the carrier of H.

We introduce h is an epimorphism as a synonym of h is epimorphism.

The following propositions are true:

(63) If h is a monomorphism and $c \in \mathrm{Im}\, h$, then $h(h^{-1}(c)) = c$.

(64) If h is a monomorphism, then $h^{-1}(h(a)) = a$.

(65) If h is a monomorphism, then h^{-1} is a homomorphism from $\mathrm{Im}\, h$ to G.

(66) h is a monomorphism iff $\mathrm{Ker}\, h = \{\mathbf{1}\}_G$.

(67) For every strict group H and for every homomorphism h from G to H holds h is an epimorphism iff $\mathrm{Im}\, h = H$.

(68) Let H be a strict group and h be a homomorphism from G to H. Suppose h is an epimorphism. Let c be an element of H. Then there exists a such that $h(a) = c$.

(69) For every normal subgroup N of G holds the canonical homomorphism onto cosets of N is an epimorphism.

Let us consider G, H, h. We say that h is isomorphism if and only if:

(Def. 14) h is an epimorphism and a monomorphism.

We introduce h is an isomorphism as a synonym of h is isomorphism.

The following propositions are true:

(70) h is an isomorphism iff $\mathrm{rng}\, h =$ the carrier of H and h is one-to-one.

(71) If h is an isomorphism, then $\mathrm{dom}\, h =$ the carrier of G and $\mathrm{rng}\, h =$ the carrier of H.

(72) Let H be a strict group and h be a homomorphism from G to H. If h is an isomorphism, then h^{-1} is a homomorphism from H to G.

(73) If h is an isomorphism and $g_1 = h^{-1}$, then g_1 is an isomorphism.

(74) If h is an isomorphism and h_1 is an isomorphism, then $h_1 \cdot h$ is an isomorphism.

(75) For every group G holds the canonical homomorphism onto cosets of $\{\mathbf{1}\}_G$ is an isomorphism.

Let us consider G, H. We say that G and H are isomorphic if and only if:

(Def. 15) There exists h which is an isomorphism.

Let us note that the predicate G and H are isomorphic is reflexive.

One can prove the following propositions:

(77)[4] For all strict groups G, H such that G and H are isomorphic holds H and G are isomorphic.

[4] The proposition (76) has been removed.

(78) If G and H are isomorphic and H and I are isomorphic, then G and I are isomorphic.

(79) If h is a monomorphism, then G and $\mathrm{Im}\,h$ are isomorphic.

(80) For all strict groups G, H such that G is trivial and H is trivial holds G and H are isomorphic.

(81) $\{\mathbf{1}\}_G$ and $\{\mathbf{1}\}_H$ are isomorphic.

(82) For every strict group G holds G and ${}^{G}/_{\{\mathbf{1}\}_G}$ are isomorphic and ${}^{G}/_{\{\mathbf{1}\}_G}$ and G are isomorphic.

(83) For every group G holds ${}^{G}/_{\Omega_G}$ is trivial.

(84) Let G, H be strict groups and h be a homomorphism from G to H. If G and H are isomorphic, then $\mathrm{Ord}(G) = \mathrm{Ord}(H)$.

(85) Let G, H be strict groups. Suppose G and H are isomorphic but G is finite or H is finite. Then G is finite and H is finite.

(86) For all strict groups G, H such that G and H are isomorphic but G is finite or H is finite holds $\mathrm{ord}(G) = \mathrm{ord}(H)$.

(87) For all strict groups G, H such that G and H are isomorphic and G is trivial holds H is trivial.

(88) Let G, H be strict groups. Suppose G and H are isomorphic but G is trivial or H is trivial. Then G is trivial and H is trivial.

(89) Let G, H be strict groups and h be a homomorphism from G to H. Suppose G and H are isomorphic but G is a commutative group or H is a commutative group. Then G is a commutative group and H is a commutative group.

(90) ${}^{G}/_{\mathrm{Ker}\,g}$ and $\mathrm{Im}\,g$ are isomorphic and $\mathrm{Im}\,g$ and ${}^{G}/_{\mathrm{Ker}\,g}$ are isomorphic.

(91) There exists a homomorphism h from ${}^{G}/_{\mathrm{Ker}\,g}$ to $\mathrm{Im}\,g$ such that h is an isomorphism and $g = h \cdot$ the canonical homomorphism onto cosets of $\mathrm{Ker}\,g$.

(92) Let M be a strict normal subgroup of G and J be a strict normal subgroup of ${}^{G}/_{M}$. Suppose $J = {}^{N}/_{(M)_N}$ and M is a subgroup of N. Then ${}^{({}^{G}/_{M})}/_{J}$ and ${}^{G}/_{N}$ are isomorphic.

(93) For every strict normal subgroup N of G holds ${}^{(B \sqcup N)}/_{(N)_{B \sqcup N}}$ and ${}^{B}/_{(B \cap N)}$ are isomorphic.

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