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Oriented Chains

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Summary. In [8] we introduced a number of notions about vertex sequences associated with undirected chains of edges in graphs. In this article, we introduce analogous concepts for oriented chains and use them to prove properties of cutting and glueing of oriented chains, and the existence of a simple oriented chain in an oriented chain.

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The articles [10], [12], [13], [4], [3], [5], [11], [1], [6], [2], [7], [8], and [9] provide the notation and terminology for this paper.

1. Oriented Vertex Sequences

For simplicity, we use the following convention: p, q are finite sequences, e, X are sets, m, n are natural numbers, G is a graph, and x, y, v, v_1, v_2, v_3, v_4 are elements of the vertices of G.

Let us consider G, let us consider x, y, and let us consider e. We say that e orientedly joins x, y if and only if:

(Def. 1) (The source of $G)(e) = x$ and (the target of $G)(e) = y$.

Next we state the proposition

(1) If e orientedly joins v_1, v_2, then e joins v_1 with v_2.

Let us consider G and let x, y be elements of the vertices of G. We say that x, y are orientedly incident if and only if:

(Def. 2) There exists a set v such that $v \in$ the edges of G and v orientedly joins x, y.

The following proposition is true

(2) If e orientedly joins v_1, v_2 and e orientedly joins v_3, v_4, then $v_1 = v_3$ and $v_2 = v_4$.

We adopt the following rules: v_5, v_6, v_7 denote finite sequences of elements of the vertices of G and c, c_1, c_2 denote oriented chains of G.

Let us consider G. One can check that there exists a chain of G which is empty and oriented.

Let us consider G, X. The functor G-SVSet$X$ yielding a set is defined as follows:

(Def. 3) $G\text{-SVSet}X = \{v : \bigvee_{e : \text{element of the edges of } G} (e \in X \wedge v = (\text{the source of } G)(e))\}$.

Let us consider G, X. The functor G-TVSet$X$ yielding a set is defined as follows:

(Def. 4) $G\text{-TVSet}X = \{v : \bigvee_{e : \text{element of the edges of } G} (e \in X \wedge v = (\text{the target of } G)(e))\}$.

We now state the proposition

(4)[1] G-SVSet $\emptyset = \emptyset$ and G-TVSet $\emptyset = \emptyset$.

Let us consider G, v_5 and let c be a finite sequence. We say that v_5 is oriented vertex seq of c if and only if:

(Def. 5) $\operatorname{len} v_5 = \operatorname{len} c + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} c$ holds $c(n)$ orientedly joins $(v_5)_n$, $(v_5)_{n+1}$.

Next we state four propositions:

(5) If v_5 is oriented vertex seq of c, then v_5 is vertex sequence of c.

(6) If v_5 is oriented vertex seq of c, then G-SVSet $\operatorname{rng} c \subseteq \operatorname{rng} v_5$.

(7) If v_5 is oriented vertex seq of c, then G-TVSet $\operatorname{rng} c \subseteq \operatorname{rng} v_5$.

(8) If $c \neq \emptyset$ and v_5 is oriented vertex seq of c, then $\operatorname{rng} v_5 \subseteq (G\text{-SVSet} \operatorname{rng} c) \cup (G\text{-TVSet} \operatorname{rng} c)$.

2. Cutting and Glueing of Oriented Chains

One can prove the following three propositions:

(9) $\langle v \rangle$ is oriented vertex seq of $\emptyset$.

(10) There exists v_5 such that v_5 is oriented vertex seq of c.

(11) If $c \neq \emptyset$ and v_6 is oriented vertex seq of c and v_7 is oriented vertex seq of c, then $v_6 = v_7$.

Let us consider G, c. Let us assume that $c \neq \emptyset$. The functor oriented-vertex-seq c yields a finite sequence of elements of the vertices of G and is defined by:

(Def. 6) oriented-vertex-seq c is oriented vertex seq of c.

One can prove the following propositions:

(12) If v_5 is oriented vertex seq of c and $c_1 = c \restriction \operatorname{Seg} n$ and $v_6 = v_5 \restriction \operatorname{Seg}(n+1)$, then v_6 is oriented vertex seq of c_1.

(13) If $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $q = \langle c(m), \ldots, c(n) \rangle$, then q is an oriented chain of G.

(14) Suppose $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ and v_5 is oriented vertex seq of c and $v_6 = \langle v_5(m), \ldots, v_5(n+1) \rangle$. Then v_6 is oriented vertex seq of c_1.

(15) Suppose v_6 is oriented vertex seq of c_1 and v_7 is oriented vertex seq of c_2 and $v_6(\operatorname{len} v_6) = v_7(1)$. Then $c_1 \frown c_2$ is an oriented chain of G.

(16) Suppose v_6 is oriented vertex seq of c_1 and v_7 is oriented vertex seq of c_2 and $v_6(\operatorname{len} v_6) = v_7(1)$ and $c = c_1 \frown c_2$ and $v_5 = v_6 \frown\frown v_7$. Then v_5 is oriented vertex seq of c.

[1] The proposition (3) has been removed.

3. Oriented Simple Chains in Oriented Chains

Let us consider G and let I_1 be an oriented chain of G. We say that I_1 is Simple if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists v_5 such that v_5 is oriented vertex seq of I_1 and for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \operatorname{len} v_5$.

Let us consider G. One can verify that there exists an oriented chain of G which is Simple.

Let us consider G. One can check that there exists a chain of G which is oriented and simple.

We now state the proposition

(18)[2] For every oriented chain q of G holds $q \restriction \operatorname{Seg} n$ is an oriented chain of G.

In the sequel s_1 denotes an oriented simple chain of G.

One can prove the following propositions:

(19) $s_1 \restriction \operatorname{Seg} n$ is an oriented simple chain of G.

(20) For every oriented chain s'_1 of G such that $s'_1 = s_1$ holds s'_1 is Simple.

(21) Every Simple oriented chain of G is an oriented simple chain of G.

(22) Suppose c is not Simple and v_5 is oriented vertex seq of c. Then there exists a FinSubsequence f_1 of c and there exists a FinSubsequence f_2 of v_5 and there exist c_1, v_6 such that $\operatorname{len} c_1 < \operatorname{len} c$ and v_6 is oriented vertex seq of c_1 and $\operatorname{len} v_6 < \operatorname{len} v_5$ and $v_5(1) = v_6(1)$ and $v_5(\operatorname{len} v_5) = v_6(\operatorname{len} v_6)$ and $\operatorname{Seq} f_1 = c_1$ and $\operatorname{Seq} f_2 = v_6$.

(23) Suppose v_5 is oriented vertex seq of c. Then there exists a FinSubsequence f_1 of c and there exists a FinSubsequence f_2 of v_5 and there exist s_1, v_6 such that $\operatorname{Seq} f_1 = s_1$ and $\operatorname{Seq} f_2 = v_6$ and v_6 is oriented vertex seq of s_1 and $v_5(1) = v_6(1)$ and $v_5(\operatorname{len} v_5) = v_6(\operatorname{len} v_6)$.

Let us consider G. Note that every oriented chain of G which is empty is also one-to-one.

One can prove the following three propositions:

(24) If p is an oriented path of G, then $p \restriction \operatorname{Seg} n$ is an oriented path of G.

(25) s_1 is an oriented path of G.

(26) Let c_1 be a finite sequence. Then

(i) c_1 is a Simple oriented chain of G iff c_1 is an oriented simple chain of G, and

(ii) if c_1 is an oriented simple chain of G, then c_1 is an oriented path of G.

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