JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Graphs

Krzysztof Hryniewiecki
Warsaw University

Summary. Definitions of graphs are introduced and their basic properties are proved. The following notions related to graph theory are introduced: subgraph, finite graph, chain and oriented chain - as a finite sequence of edges, path and oriented path - as a finite sequence of different edges, cycle and oriented cycle, incidency of graph's vertices, a sum of two graphs, a degree of a vertice, a set of all subgraphs of a graph. Many ideas of this article have been taken from [11].

MML Identifier: `GRAPH_1`.

WWW: `http://mizar.org/JFM/Vol2/graph_1.html`

The articles [9], [7], [10], [12], [4], [5], [3], [8], [6], [1], and [2] provide the notation and terminology for this paper.

We use the following convention: x, y, v are sets and n, m are natural numbers.

We consider multi graph structures as systems

$\langle$ vertices, edges, a source, a target $\rangle$,

where the vertices and the edges constitute sets and the source and the target are functions from the edges into the vertices.

Let I_1 be a multi graph structure. We say that I_1 is graph-like if and only if:

(Def. 1) The vertices of I_1 are a non empty set.

Let us note that there exists a multi graph structure which is strict and graph-like.

A graph is a graph-like multi graph structure.

In the sequel G, G_1, G_2, G_3 denote graphs.

Let us consider G_1, G_2. Let us assume that the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2. The functor $G_1 \cup G_2$ yielding a strict graph is defined by the conditions (Def. 2).

(Def. 2)(i) The vertices of $G_1 \cup G_2 =$ (the vertices of G_1) $\cup$ (the vertices of G_2),

(ii) the edges of $G_1 \cup G_2 =$ (the edges of G_1) $\cup$ (the edges of G_2),

(iii) for every v such that $v \in$ the edges of G_1 holds (the source of $G_1 \cup G_2)(v) =$ (the source of $G_1)(v)$ and (the target of $G_1 \cup G_2)(v) =$ (the target of $G_1)(v)$, and

(iv) for every v such that $v \in$ the edges of G_2 holds (the source of $G_1 \cup G_2)(v) =$ (the source of $G_2)(v)$ and (the target of $G_1 \cup G_2)(v) =$ (the target of $G_2)(v)$.

Let G, G_1, G_2 be graphs. We say that G is a sum of G_1 and G_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) The target of $G_1 \approx$ the target of G_2,

(ii) the source of $G_1 \approx$ the source of G_2, and

(iii) the multi graph structure of $G = G_1 \cup G_2$.

Let I_1 be a graph. We say that I_1 is oriented if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let given x, y. Suppose that

(i) $x \in$ the edges of I_1,

(ii) $y \in$ the edges of I_1,

(iii) (the source of $I_1)(x) =$ (the source of $I_1)(y)$, and

(iv) (the target of $I_1)(x) =$ (the target of $I_1)(y)$.

Then $x = y$.

We say that I_1 is non-multi if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let given x, y. Suppose that

(i) $x \in$ the edges of I_1,

(ii) $y \in$ the edges of I_1, and

(iii) (the source of $I_1)(x) =$ (the source of $I_1)(y)$ and (the target of $I_1)(x) =$ (the target of $I_1)(y)$ or (the source of $I_1)(x) =$ (the target of $I_1)(y)$ and (the source of $I_1)(y) =$ (the target of $I_1)(x)$.

Then $x = y$.

We say that I_1 is simple if and only if:

(Def. 6) It is not true that there exists x such that $x \in$ the edges of I_1 and (the source of $I_1)(x) =$ (the target of $I_1)(x)$.

We say that I_1 is connected if and only if:

(Def. 7) It is not true that there exist graphs G_1, G_2 such that the vertices of G_1 misses the vertices of G_2 and I_1 is a sum of G_1 and G_2.

Let I_1 be a multi graph structure. We say that I_1 is finite if and only if:

(Def. 8) The vertices of I_1 are finite and the edges of I_1 are finite.

One can verify that there exists a multi graph structure which is finite and there exists a graph which is finite, non-multi, oriented, simple, and connected.

In the sequel x, y denote elements of the vertices of G.

Let us consider G, let us consider x, y, and let us consider v. We say that v joins x with y if and only if:

(Def. 9) (The source of $G)(v) = x$ and (the target of $G)(v) = y$ or (the source of $G)(v) = y$ and (the target of $G)(v) = x$.

Let us consider G and let x, y be elements of the vertices of G. We say that x and y are incident if and only if:

(Def. 10) There exists a set v such that $v \in$ the edges of G and v joins x with y.

Let G be a graph. A finite sequence is called a chain of G if it satisfies the conditions (Def. 11).

(Def. 11)(i) For every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ holds $\mathrm{it}(n) \in$ the edges of G, and

(ii) there exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathrm{it} + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G and for every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $\mathrm{it}(n)$ joins x' with y'.

Let G be a graph. We see that the chain of G is a finite sequence of elements of the edges of G.

Let G be a graph and let I_1 be a chain of G. We say that I_1 is oriented if and only if:

(Def. 12) For every n such that $1 \leq n$ and $n < \operatorname{len} I_1$ holds (the source of $G)(I_1(n+1)) =$ (the target of $G)(I_1(n))$.

Let G be a graph. Note that there exists a chain of G which is oriented.

Let G be a graph and let I_1 be a chain of G. Let us observe that I_1 is one-to-one if and only if:

(Def. 13) For all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} I_1$ holds $I_1(n) \neq I_1(m)$.

Let G be a graph. One can check that there exists a chain of G which is one-to-one.

Let G be a graph. A path of G is an one-to-one chain of G.

Let G be a graph. Note that there exists a chain of G which is one-to-one and oriented.

Let G be a graph. An oriented path of G is an one-to-one oriented chain of G.

Let G be a graph and let I_1 be a path of G. We say that I_1 is cyclic if and only if the condition (Def. 15) is satisfied.

(Def. 15)[1] There exists a finite sequence p such that

(i) $\operatorname{len} p = \operatorname{len} I_1 + 1$,

(ii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G,

(iii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} I_1$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $I_1(n)$ joins x' with y', and

(iv) $p(1) = p(\operatorname{len} p)$.

Let G be a graph. One can verify that there exists a path of G which is cyclic.

Let G be a graph. A cycle of G is a cyclic path of G.

Let G be a graph. Observe that there exists an oriented path of G which is cyclic.

Let G be a graph. An oriented cycle of G is a cyclic oriented path of G.

Let G be a graph. A graph is called a subgraph of G if it satisfies the conditions (Def. 17).

(Def. 17)[2](i) The vertices of it $\subseteq$ the vertices of G,

(ii) the edges of it $\subseteq$ the edges of G, and

(iii) for every v such that $v \in$ the edges of it holds (the source of it)$(v) =$ (the source of $G)(v)$ and (the target of it)$(v) =$ (the target of $G)(v)$ and (the source of $G)(v) \in$ the vertices of it and (the target of $G)(v) \in$ the vertices of it.

Let G be a graph. Observe that there exists a subgraph of G which is strict.

Let G be a finite graph. The number of vertices of G yielding a natural number is defined as follows:

(Def. 18) There exists a finite set B such that $B =$ the vertices of G and the number of vertices of $G = \operatorname{card} B$.

The number of edges of G yielding a natural number is defined by:

(Def. 19) There exists a finite set B such that $B =$ the edges of G and the number of edges of $G = \operatorname{card} B$.

Let G be a finite graph and let x be an element of the vertices of G. The functor $\operatorname{EdgIn}(x)$ yielding a natural number is defined by the condition (Def. 20).

(Def. 20) There exists a finite set X such that for every set z holds $z \in X$ iff $z \in$ the edges of G and (the target of $G)(z) = x$ and $\operatorname{EdgIn}(x) = \operatorname{card} X$.

The functor $\operatorname{EdgOut}(x)$ yielding a natural number is defined by the condition (Def. 21).

(Def. 21) There exists a finite set X such that for every set z holds $z \in X$ iff $z \in$ the edges of G and (the source of $G)(z) = x$ and $\operatorname{EdgOut}(x) = \operatorname{card} X$.

Let G be a finite graph and let x be an element of the vertices of G. The degree of x yields a natural number and is defined as follows:

[1] The definition (Def. 14) has been removed.

[2] The definition (Def. 16) has been removed.

(Def. 22) The degree of $x = \mathrm{EdgIn}(x) + \mathrm{EdgOut}(x)$.

Let G_1, G_2 be graphs. The predicate $G_1 \subseteq G_2$ is defined as follows:

(Def. 23) G_1 is a subgraph of G_2.

Let us note that the predicate $G_1 \subseteq G_2$ is reflexive.

Let G be a graph. The functor 2^G yields a set and is defined by:

(Def. 24) For every set x holds $x \in 2^G$ iff x is a strict subgraph of G.

The scheme *GraphSeparation* deals with a graph $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a set X such that for every set x holds $x \in X$ iff x is a strict subgraph of $\mathcal{A}$ and $\mathcal{P}[x]$

for all values of the parameters.

One can prove the following propositions:

(1) Let G be a graph. Then

(i) dom(the source of G) = the edges of G,

(ii) dom(the target of G) = the edges of G,

(iii) rng(the source of G) $\subseteq$ the vertices of G, and

(iv) rng(the target of G) $\subseteq$ the vertices of G.

(2) For every element x of the vertices of G holds $x \in$ the vertices of G.

(3) Let v be a set. Suppose $v \in$ the edges of G. Then (the source of G)$(v) \in$ the vertices of G and (the target of G)$(v) \in$ the vertices of G.

(4) For every chain p of G holds $p \restriction \mathrm{Seg}\, n$ is a chain of G.

(5) If $G_1 \subseteq G$, then the source of $G_1 \subseteq$ the source of G and the target of $G_1 \subseteq$ the target of G.

(6) Suppose the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2. Then

(i) the source of $G_1 \cup G_2$ = (the source of G_1) $\cup$ (the source of G_2), and

(ii) the target of $G_1 \cup G_2$ = (the target of G_1) $\cup$ (the target of G_2).

(7) For every strict graph G holds $G = G \cup G$.

(8) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \cup G_2 = G_2 \cup G_1$.

(9) Suppose that

(i) the source of $G_1 \approx$ the source of G_2,

(ii) the target of $G_1 \approx$ the target of G_2,

(iii) the source of $G_1 \approx$ the source of G_3,

(iv) the target of $G_1 \approx$ the target of G_3,

(v) the source of $G_2 \approx$ the source of G_3, and

(vi) the target of $G_2 \approx$ the target of G_3.

Then $(G_1 \cup G_2) \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(10) If G is a sum of G_1 and G_2, then G is a sum of G_2 and G_1.

(11) Every strict graph G is a sum of G and G.

(12) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 = G_2 \cup G_1$.

(13) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$ and $G_3 \subseteq G$, then $(G_1 \cup G_2) \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(14) $\emptyset$ is a cyclic oriented path of G.

(15) Let H_1, H_2 be strict subgraphs of G. Suppose the vertices of $H_1 =$ the vertices of H_2 and the edges of $H_1 =$ the edges of H_2. Then $H_1 = H_2$.

(16) For all strict graphs G_1, G_2 such that $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$ holds $G_1 = G_2$.

(17) If $G_1 \subseteq G_2$ and $G_2 \subseteq G_3$, then $G_1 \subseteq G_3$.

(18) If G is a sum of G_1 and G_2, then $G_1 \subseteq G$ and $G_2 \subseteq G$.

(19) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(20) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(21) If $G_1 \subseteq G_3$ and $G_2 \subseteq G_3$ and G is a sum of G_1 and G_2, then $G \subseteq G_3$.

(22) If $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 \subseteq G$.

(23) For all strict graphs G_1, G_2 such that $G_1 \subseteq G_2$ holds $G_1 \cup G_2 = G_2$ and $G_2 \cup G_1 = G_2$.

(24) Suppose the source of $G_1 \approx$ the source of G_2 but the target of $G_1 \approx$ the target of G_2 but $G_1 \cup G_2 = G_2$ or $G_2 \cup G_1 = G_2$. Then $G_1 \subseteq G_2$.

(27)[3] For every oriented graph G such that $G_1 \subseteq G$ holds G_1 is oriented.

(28) For every non-multi graph G such that $G_1 \subseteq G$ holds G_1 is non-multi.

(29) For every simple graph G such that $G_1 \subseteq G$ holds G_1 is simple.

(30) For every strict graph G_1 holds $G_1 \in 2^G$ iff $G_1 \subseteq G$.

(31) For every strict graph G holds $G \in 2^G$.

(32) For every strict graph G_1 holds $G_1 \subseteq G_2$ iff $2^{G_1} \subseteq 2^{G_2}$.

(34)[4] For every strict graph G holds $\{G\} \subseteq 2^G$.

(35) Let G_1, G_2 be strict graphs. Suppose the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 and $2^{G_1 \cup G_2} \subseteq 2^{G_1} \cup 2^{G_2}$. Then $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$.

(36) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $2^{G_1} \cup 2^{G_2} \subseteq 2^{G_1 \cup G_2}$.

(37) If $G_1 \in 2^G$ and $G_2 \in 2^G$, then $G_1 \cup G_2 \in 2^G$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

---

[3] The propositions (25) and (26) have been removed.
[4] The proposition (33) has been removed.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[11] Robin Wilson. *Wprowadzenie do teorii grafów*. PWN, 1985.

[12] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received December 5, 1990

Published January 2, 2004