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Basic Properties of Fuzzy Set Operation and Membership Function

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Summary. This article introduces the fuzzy theory. The definition of the difference set, algebraic product and algebraic sum of fuzzy set is shown. In addition, basic properties of those operations are described. Basic properties of fuzzy set are a little different from those of crisp set.

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The articles [5], [1], [8], [6], [7], [2], [9], [3], and [4] provide the notation and terminology for this paper.

1. Basic Properties of Membership Function and Difference Set

For simplicity, we use the following convention: C denotes a non empty set, c denotes an element of C, f, h, g, h_1 denote membership functions of C, A denotes a FuzzySet of C, f, B denotes a FuzzySet of C, g, D denotes a FuzzySet of C, h, D_1 denotes a FuzzySet of C, h_1, X denotes a Universal FuzzySet of C, and E denotes an Empty FuzzySet of C.

Next we state four propositions:

(1) For every element x of C and for every membership function h of C holds $0 \leq h(x)$ and $h(x) \leq 1$.

(2) For every element x of C holds $(\mathrm{EMF}C)(x) = 0$ and $(\mathrm{UMF}C)(x) = 1$.

(3) For every c such that $f(c) \leq h(c)$ holds $(\max(f, \min(g, h)))(c) = (\min(\max(f, g), h))(c)$.

(4) If $A \subseteq D$, then $A \cup B \cap D = (A \cup B) \cap D$.

Let C be a non empty set, let f, g be membership functions of C, let A be a FuzzySet of C, f, and let B be a FuzzySet of C, g. The functor $A \setminus B$ yields a FuzzySet of C, $\min(f, 1\text{-minus}\, g)$ and is defined as follows:

(Def. 1) $A \setminus B = [\!: C, (\min(f, 1\text{-minus}\, g))^{\circ} C :\!]$.

We now state a number of propositions:

(6)[1] $1\text{-minus} \min(f, 1\text{-minus}\, g) = \max(1\text{-minus}\, f, g)$.

(7) $(A \setminus B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B$.

[1] The proposition (5) has been removed.

(8) For every c such that $f(c) \leq g(c)$ holds $(\min(f, 1\text{-minus}\, h))(c) \leq (\min(g, 1\text{-minus}\, h))(c)$.

(9) If $A \subseteq B$, then $A \setminus D \subseteq B \setminus D$.

(10) For every c such that $f(c) \leq g(c)$ holds $(\min(h, 1\text{-minus}\, g))(c) \leq (\min(h, 1\text{-minus}\, f))(c)$.

(11) If $A \subseteq B$, then $D \setminus B \subseteq D \setminus A$.

(12) For every c such that $f(c) \leq g(c)$ and $h(c) \leq h_1(c)$ holds $(\min(f, 1\text{-minus}\, h_1))(c) \leq (\min(g, 1\text{-minus}\, h))(c)$.

(13) If $A \subseteq B$ and $D \subseteq D_1$, then $A \setminus D_1 \subseteq B \setminus D$.

(14) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leq f(c)$.

(15) $A \setminus B \subseteq A$.

(16) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leq (\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(17) $A \setminus B \subseteq A \dot{-} B$.

(18) $A \setminus E = A$.

(19) $E \setminus A = E$.

(20) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leq (\min(f, 1\text{-minus}\, \min(f, g)))(c)$.

(21) $A \setminus B \subseteq A \setminus A \cap B$.

(22) For every c holds $(\max(\min(f, g), \min(f, 1\text{-minus}\, g)))(c) \leq f(c)$.

(23) For every c holds $(\max(f, \min(g, 1\text{-minus}\, f)))(c) \leq (\max(f, g))(c)$.

(24) $A \cup (B \setminus A) \subseteq A \cup B$.

(25) $A \cap B \cup (A \setminus B) \subseteq A$.

(26) $\min(f, 1\text{-minus}\, \min(g, 1\text{-minus}\, h)) = \max(\min(f, 1\text{-minus}\, g), \min(f, h))$.

(27) $A \setminus (B \setminus D) = (A \setminus B) \cup A \cap D$.

(28) For every c holds $(\min(f, g))(c) \leq (\min(f, 1\text{-minus}\, \min(f, 1\text{-minus}\, g)))(c)$.

(29) $A \cap B \subseteq A \setminus (A \setminus B)$.

(30) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leq (\min(\max(f, g), 1\text{-minus}\, g))(c)$.

(31) $A \setminus B \subseteq (A \cup B) \setminus B$.

(32) $\min(f, 1\text{-minus}\, \max(g, h)) = \min(\min(f, 1\text{-minus}\, g), \min(f, 1\text{-minus}\, h))$.

(33) $A \setminus (B \cup D) = (A \setminus B) \cap (A \setminus D)$.

(34) $\min(f, 1\text{-minus}\, \min(g, h)) = \max(\min(f, 1\text{-minus}\, g), \min(f, 1\text{-minus}\, h))$.

(35) $A \setminus B \cap D = (A \setminus B) \cup (A \setminus D)$.

(36) $\min(\min(f, 1\text{-minus}\, g), 1\text{-minus}\, h) = \min(f, 1\text{-minus}\, \max(g, h))$.

(37) $A \setminus B \setminus D = A \setminus (B \cup D)$.

(38) For every c holds $(\min(\max(f, g), 1\text{-minus}\, \min(f, g)))(c) \geq (\max(\min(f, 1\text{-minus}\, g), \min(g, 1\text{-minus}\, f)))(c)$.

(39) $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus A \cap B$.

(40) $\min(\max(f, g), 1\text{-minus}\, h) = \max(\min(f, 1\text{-minus}\, h), \min(g, 1\text{-minus}\, h))$.

(41) $(A \cup B) \setminus D = (A \setminus D) \cup (B \setminus D)$.

(42) For every c such that $(\min(f, \text{1-minus}\, g))(c) \leq h(c)$ and $(\min(g, \text{1-minus}\, f))(c) \leq h(c)$ holds $(\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c) \leq h(c)$.

(43) If $A \setminus B \subseteq D$ and $B \setminus A \subseteq D$, then $A \dot{-} B \subseteq D$.

(44) $A \cap (B \setminus D) = A \cap B \setminus D$.

(45) For every c holds $(\min(f, \min(g, \text{1-minus}\, h)))(c) \leq (\min(\min(f, g), \text{1-minus}\, \min(f, h)))(c)$.

(46) $A \cap (B \setminus D) \subseteq A \cap B \setminus A \cap D$.

(47) For every c holds $(\min(\max(f, g), \text{1-minus}\, \min(f, g)))(c) \geq (\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c)$.

(48) $A \dot{-} B \subseteq (A \cup B) \setminus A \cap B$.

(49) For every c holds $(\max(\min(f, g), \text{1-minus}\, \max(f, g)))(c) \leq (\text{1-minus}\, \max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c$

(50) $A \cap B \cup (A \cup B)^{\rm c} \subseteq (A \dot{-} B)^{\rm c}$.

(51) $\min(\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)), \text{1-minus}\, h) = \max(\min(f, \text{1-minus}\, \max(g, h)), \min(g, \text{1-minus}\, \max($

(52) $(A \dot{-} B) \setminus D = (A \setminus (B \cup D)) \cup (B \setminus (A \cup D))$.

(53) For every c holds $(\min(f, \text{1-minus}\, \max(\min(g, \text{1-minus}\, h), \min(\text{1-minus}\, g, h))))(c) \geq (\max(\min(f, \text{1-minus}\, \max(g, h)), \min(\min(f, g), h)))(c)$.

(54) $(A \setminus (B \cup D)) \cup A \cap B \cap D \subseteq A \setminus (B \dot{-} D)$.

(55) For every c such that $f(c) \leq g(c)$ holds $g(c) \geq (\max(f, \min(g, \text{1-minus}\, f)))(c)$.

(56) If $A \subseteq B$, then $A \cup (B \setminus A) \subseteq B$.

(57) For every c holds $(\max(f, g))(c) \geq (\max(\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)), \min(f, g)))(c)$.

(59)[2] If $\min(f, \text{1-minus}\, g) = \text{EMF}\, C$, then for every c holds $f(c) \leq g(c)$.

(60) If $A \setminus B = E$, then $A \subseteq B$.

(61) If $\min(f, g) = \text{EMF}\, C$, then $\min(f, \text{1-minus}\, g) = f$.

(62) If $A \cap B = E$, then $A \setminus B = A$.

2. Algebraic Product and Algebraic Sum

Let C be a non empty set and let h, g be membership functions of C. The functor $h \cdot g$ yields a membership function of C and is defined as follows:

(Def. 2) For every element c of C holds $(h \cdot g)(c) = h(c) \cdot g(c)$.

Let C be a non empty set and let h, g be membership functions of C. The functor $h \oplus g$ yields a membership function of C and is defined by:

(Def. 3) For every element c of C holds $(h \oplus g)(c) = (h(c) + g(c)) - h(c) \cdot g(c)$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \cdot B$ yields a FuzzySet of C, $h \cdot g$ and is defined by:

(Def. 4) $A \cdot B = [\!: C, (h \cdot g)^{\circ} C :\!]$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \oplus B$ yields a FuzzySet of C, $h \oplus g$ and is defined by:

[2] The proposition (58) has been removed.

(Def. 5) $A \oplus B = [\!: C, (h \oplus g)^\circ C :\!]$.

One can prove the following propositions:

(63) For every c holds $(f \cdot f)(c) \leq f(c)$ and $(f \oplus f)(c) \geq f(c)$.

(64) $A \cdot A \subseteq A$ and $A \subseteq A \oplus A$.

(65) $f \cdot g = g \cdot f$ and $f \oplus g = g \oplus f$.

(66) $A \cdot B = B \cdot A$ and $A \oplus B = B \oplus A$.

(67) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

(68) $(A \cdot B) \cdot D = A \cdot (B \cdot D)$.

(69) $(f \oplus g) \oplus h = f \oplus (g \oplus h)$.

(70) $(A \oplus B) \oplus D = A \oplus (B \oplus D)$.

(71) For every c holds $(f \cdot (f \oplus g))(c) \leq f(c)$ and $(f \oplus f \cdot g)(c) \geq f(c)$.

(72) $A \cdot (A \oplus B) \subseteq A$ and $A \subseteq A \oplus A \cdot B$.

(73) For every c holds $(f \cdot (g \oplus h))(c) \leq (f \cdot g \oplus f \cdot h)(c)$.

(74) $A \cdot (B \oplus D) \subseteq A \cdot B \oplus A \cdot D$.

(75) For every c holds $((f \oplus g) \cdot (f \oplus h))(c) \leq (f \oplus g \cdot h)(c)$.

(76) $(A \oplus B) \cdot (A \oplus D) \subseteq A \oplus B \cdot D$.

(77) 1-minus $f \cdot g =$ 1-minus $f \oplus$ 1-minus g.

(78) $(A \cdot B)^{\mathrm{c}} = A^{\mathrm{c}} \oplus B^{\mathrm{c}}$.

(79) 1-minus $f \oplus g =$ 1-minus $f \cdot$ 1-minus g.

(80) $(A \oplus B)^{\mathrm{c}} = A^{\mathrm{c}} \cdot B^{\mathrm{c}}$.

(81) $f \oplus g =$ 1-minus 1-minus $f \cdot$ 1-minus g.

(82) $A \oplus B = (A^{\mathrm{c}} \cdot B^{\mathrm{c}})^{\mathrm{c}}$.

(83) $f \cdot \mathrm{EMF}C = \mathrm{EMF}C$ and $f \cdot \mathrm{UMF}C = f$.

(84) $A \cdot E = E$ and $A \cdot X = A$.

(85) $f \oplus \mathrm{EMF}C = f$ and $f \oplus \mathrm{UMF}C = \mathrm{UMF}C$.

(86) $A \oplus E = A$ and $A \oplus X = X$.

(89)[3] $E \subseteq A \cdot A^{\mathrm{c}}$ and $A \oplus A^{\mathrm{c}} \subseteq X$.

(90) For every c holds $(f \cdot g)(c) \leq (\min(f, g))(c)$.

(91) $A \cdot B \subseteq A \cap B$.

(92) For every c holds $(\max(f, g))(c) \leq (f \oplus g)(c)$.

(93) $A \cup B \subseteq A \oplus B$.

(94) For all real numbers a, b, c such that $0 \leq c$ holds $c \cdot \max(a, b) = \max(c \cdot a, c \cdot b)$ and $c \cdot \min(a, b) = \min(c \cdot a, c \cdot b)$.

[3] The propositions (87) and (88) have been removed.

(95) For all real numbers a, b, c holds $c+\max(a,b)=\max(c+a,c+b)$ and $c+\min(a,b)=\min(c+a,c+b)$.

(96) $f\cdot\max(g,h)=\max(f\cdot g, f\cdot h)$.

(97) $f\cdot\min(g,h)=\min(f\cdot g, f\cdot h)$.

(98) $A\cdot(B\cap D)=(A\cdot B)\cap(A\cdot D)$ and $A\cdot(B\cup D)=A\cdot B\cup A\cdot D$.

(99) $f\oplus\max(g,h)=\max(f\oplus g, f\oplus h)$.

(100) $f\oplus\min(g,h)=\min(f\oplus g, f\oplus h)$.

(101) $A\oplus(B\cup D)=(A\oplus B)\cup(A\oplus D)$ and $A\oplus B\cap D=(A\oplus B)\cap(A\oplus D)$.

(102) For every c holds $(\max(f,g)\cdot\max(f,h))(c)\leq(\max(f,g\cdot h))(c)$.

(103) For every c holds $(\min(f,g)\cdot\min(f,h))(c)\leq(\min(f,g\cdot h))(c)$.

(104) $(A\cup B)\cdot(A\cup D)\subseteq A\cup B\cdot D$ and $(A\cap B)\cdot(A\cap D)\subseteq A\cap(B\cdot D)$.

(105) For every element c of C and for all membership functions f, g of C holds $(f\oplus g)(c)=1-(1-f(c))\cdot(1-g(c))$.

(106) For every c holds $(\max(f,g\oplus h))(c)\leq(\max(f,g)\oplus\max(f,h))(c)$.

(107) For every c holds $(\min(f,g\oplus h))(c)\leq(\min(f,g)\oplus\min(f,h))(c)$.

(108) $A\cup(B\oplus D)\subseteq(A\cup B)\oplus(A\cup D)$ and $A\cap(B\oplus D)\subseteq A\cap B\oplus A\cap D$.

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