

The Composition of Functors and Transformations in Alternative Categories

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The articles [10], [5], [16], [2], [4], [3], [1], [9], [11], [12], [6], [13], [14], [7], [15], and [8] provide the notation and terminology for this paper.

1. PRELIMINARIES

One can check that there exists a non empty category structure which is transitive, associative, and strict and has units.

Let A be a non empty transitive category structure and let B be a non empty category structure with units. Observe that there exists a functor structure from A to B which is strict, comp-preserving, comp-reversing, precovariant, precontravariant, and feasible.

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. One can check that there exists a functor structure from A to B which is strict, comp-preserving, comp-reversing, precovariant, precontravariant, feasible, and id-preserving.

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. Note that there exists a functor from A to B which is strict, feasible, covariant, and contravariant.

We now state several propositions:

- (1) Let C be a category, o_1, o_2, o_3, o_4 be objects of C , a be a morphism from o_1 to o_2 , b be a morphism from o_2 to o_3 , c be a morphism from o_1 to o_4 , and d be a morphism from o_4 to o_3 . Suppose $b \cdot a = d \cdot c$ and $a \cdot a^{-1} = \text{id}_{(o_2)}$ and $d^{-1} \cdot d = \text{id}_{(o_4)}$ and $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_1 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$ and $\langle o_3, o_4 \rangle \neq \emptyset$ and $\langle o_4, o_3 \rangle \neq \emptyset$. Then $c \cdot a^{-1} = d^{-1} \cdot b$.
- (2) Let A be a non empty transitive category structure, B, C be non empty category structures with units, F be a feasible precovariant functor structure from A to B , G be a functor structure from B to C , and o, o_1 be objects of A . Then $\text{Morph-Map}_{G \cdot F}(o, o_1) = \text{Morph-Map}_G(F(o), F(o_1)) \cdot \text{Morph-Map}_F(o, o_1)$.
- (3) Let A be a non empty transitive category structure, B, C be non empty category structures with units, F be a feasible precontravariant functor structure from A to B , G be a functor structure from B to C , and o, o_1 be objects of A . Then $\text{Morph-Map}_{G \cdot F}(o, o_1) = \text{Morph-Map}_G(F(o_1), F(o)) \cdot \text{Morph-Map}_F(o, o_1)$.

- (4) Let A be a non empty transitive category structure, B be a non empty category structure with units, and F be a feasible functor structure from A to B . Then $\text{id}_B \cdot F$ = the functor structure of F .
- (5) Let A be a transitive non empty category structure with units, B be a non empty category structure with units, and F be a feasible functor structure from A to B . Then $F \cdot \text{id}_A$ = the functor structure of F .

For simplicity, we adopt the following convention: A denotes a non empty category structure, B, C denote non empty reflexive category structures, F denotes a feasible precovariant functor structure from A to B , G denotes a feasible precovariant functor structure from B to C , M denotes a feasible precontravariant functor structure from A to B , N denotes a feasible precontravariant functor structure from B to C , o_1, o_2 denote objects of A , and m denotes a morphism from o_1 to o_2 .

We now state four propositions:

- (6) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(G \cdot F)(m) = G(F(m))$.
- (7) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(N \cdot M)(m) = N(M(m))$.
- (8) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(N \cdot F)(m) = N(F(m))$.
- (9) If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(G \cdot M)(m) = G(M(m))$.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precovariant comp-preserving functor structure from A to B , and let G be a feasible precovariant comp-preserving functor structure from B to C . One can verify that $G \cdot F$ is comp-preserving.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precontravariant comp-reversing functor structure from A to B , and let G be a feasible precontravariant comp-reversing functor structure from B to C . Note that $G \cdot F$ is comp-preserving.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precovariant comp-preserving functor structure from A to B , and let G be a feasible precontravariant comp-reversing functor structure from B to C . Observe that $G \cdot F$ is comp-reversing.

Let A be a non empty transitive category structure, let B be a transitive non empty category structure with units, let C be a non empty category structure with units, let F be a feasible precontravariant comp-reversing functor structure from A to B , and let G be a feasible precovariant comp-preserving functor structure from B to C . One can check that $G \cdot F$ is comp-reversing.

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a covariant functor from A to B , and let G be a covariant functor from B to C . Then $G \cdot F$ is a strict covariant functor from A to C .

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a contravariant functor from A to B , and let G be a contravariant functor from B to C . Then $G \cdot F$ is a strict covariant functor from A to C .

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a covariant functor from A to B , and let G be a contravariant functor from B to C . Then $G \cdot F$ is a strict contravariant functor from A to C .

Let A, B be transitive non empty category structures with units, let C be a non empty category structure with units, let F be a contravariant functor from A to B , and let G be a covariant functor from B to C . Then $G \cdot F$ is a strict contravariant functor from A to C .

For simplicity, we adopt the following convention: A, B, C, D are transitive non empty category structures with units, F_1, F_2, F_3 are covariant functors from A to B , G_1, G_2, G_3 are covariant functors from B to C , H_1, H_2 are covariant functors from C to D , p is a transformation from F_1 to F_2 , p_1 is a transformation from F_2 to F_3 , q is a transformation from G_1 to G_2 , q_1 is a transformation from G_2 to G_3 , and r is a transformation from H_1 to H_2 .

We now state the proposition

- (10) If F_1 is transformable to F_2 and G_1 is transformable to G_2 , then $G_1 \cdot F_1$ is transformable to $G_2 \cdot F_2$.

2. THE COMPOSITION OF FUNCTORS WITH TRANSFORMATIONS

Let A, B, C be transitive non empty category structures with units, let F_1, F_2 be covariant functors from A to B , let t be a transformation from F_1 to F_2 , and let G be a covariant functor from B to C . Let us assume that F_1 is transformable to F_2 . The functor $G \cdot t$ yielding a transformation from $G \cdot F_1$ to $G \cdot F_2$ is defined by:

(Def. 1) For every object o of A holds $(G \cdot t)(o) = G(t[o])$.

We now state the proposition

(11) For every object o of A such that F_1 is transformable to F_2 holds $(G_1 \cdot p)[o] = G_1(p[o])$.

Let A, B, C be transitive non empty category structures with units, let G_1, G_2 be covariant functors from B to C , let F be a covariant functor from A to B , and let s be a transformation from G_1 to G_2 . Let us assume that G_1 is transformable to G_2 . The functor $s \cdot F$ yields a transformation from $G_1 \cdot F$ to $G_2 \cdot F$ and is defined by:

(Def. 2) For every object o of A holds $(s \cdot F)(o) = s[F(o)]$.

We now state a number of propositions:

- (12) For every object o of A such that G_1 is transformable to G_2 holds $(q \cdot F_1)[o] = q[F_1(o)]$.
- (13) If F_1 is transformable to F_2 and F_2 is transformable to F_3 , then $G_1 \cdot (p_1 \circ p) = G_1 \cdot p_1 \circ G_1 \cdot p$.
- (14) If G_1 is transformable to G_2 and G_2 is transformable to G_3 , then $(q_1 \circ q) \cdot F_1 = q_1 \cdot F_1 \circ q \cdot F_1$.
- (15) If H_1 is transformable to H_2 , then $(r \cdot G_1) \cdot F_1 = r \cdot (G_1 \cdot F_1)$.
- (16) If G_1 is transformable to G_2 , then $(H_1 \cdot q) \cdot F_1 = H_1 \cdot (q \cdot F_1)$.
- (17) If F_1 is transformable to F_2 , then $(H_1 \cdot G_1) \cdot p = H_1 \cdot (G_1 \cdot p)$.
- (18) $\text{id}_{(G_1)} \cdot F_1 = \text{id}_{G_1 \cdot F_1}$.
- (19) $G_1 \cdot \text{id}_{(F_1)} = \text{id}_{G_1 \cdot F_1}$.
- (20) If F_1 is transformable to F_2 , then $\text{id}_B \cdot p = p$.
- (21) If G_1 is transformable to G_2 , then $q \cdot \text{id}_B = q$.

3. THE COMPOSITION OF TRANSFORMATIONS

Let A, B, C be transitive non empty category structures with units, let F_1, F_2 be covariant functors from A to B , let G_1, G_2 be covariant functors from B to C , let t be a transformation from F_1 to F_2 , and let s be a transformation from G_1 to G_2 . The functor $s t$ yields a transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$ and is defined by:

(Def. 3) $s t = s \cdot F_2 \circ G_1 \cdot t$.

Next we state several propositions:

- (22) Let q be a natural transformation from G_1 to G_2 . Suppose F_1 is transformable to F_2 and G_1 is naturally transformable to G_2 . Then $q p = G_2 \cdot p \circ q \cdot F_1$.
- (23) If F_1 is transformable to F_2 , then $\text{id}_{\text{id}_B} p = p$.
- (24) If G_1 is transformable to G_2 , then $q \text{id}_{\text{id}_B} = q$.
- (25) If F_1 is transformable to F_2 , then $G_1 \cdot p = \text{id}_{(G_1)} p$.
- (26) If G_1 is transformable to G_2 , then $q \cdot F_1 = q \text{id}_{(F_1)}$.

We adopt the following convention: A, B, C, D are categories, F_1, F_2, F_3 are covariant functors from A to B , and G_1, G_2, G_3 are covariant functors from B to C .

Next we state the proposition

- (27) Let H_1, H_2 be covariant functors from C to D , t be a transformation from F_1 to F_2 , s be a transformation from G_1 to G_2 , and u be a transformation from H_1 to H_2 . Suppose F_1 is transformable to F_2 and G_1 is transformable to G_2 and H_1 is transformable to H_2 . Then $(u s) t = u (s t)$.

In the sequel t denotes a natural transformation from F_1 to F_2 , s denotes a natural transformation from G_1 to G_2 , and s_1 denotes a natural transformation from G_2 to G_3 .

We now state four propositions:

- (28) If F_1 is naturally transformable to F_2 , then $G_1 \cdot t$ is a natural transformation from $G_1 \cdot F_1$ to $G_1 \cdot F_2$.
- (29) If G_1 is naturally transformable to G_2 , then $s \cdot F_1$ is a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_1$.
- (30) Suppose F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 . Then $G_1 \cdot F_1$ is naturally transformable to $G_2 \cdot F_2$ and $s t$ is a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$.
- (31) Let t be a transformation from F_1 to F_2 and t_1 be a transformation from F_2 to F_3 . Suppose that
- (i) F_1 is naturally transformable to F_2 ,
 - (ii) F_2 is naturally transformable to F_3 ,
 - (iii) G_1 is naturally transformable to G_2 , and
 - (iv) G_2 is naturally transformable to G_3 .

Then $(s_1 \circ s) (t_1 \circ t) = s_1 t_1 \circ s t$.

4. NATURAL EQUIVALENCES

We now state the proposition

- (32) Suppose F_1 is naturally transformable to F_2 and F_2 is transformable to F_1 and for every object a of A holds $t[a]$ is iso. Then
- (i) F_2 is naturally transformable to F_1 , and
 - (ii) there exists a natural transformation f from F_2 to F_1 such that for every object a of A holds $f(a) = t[a]^{-1}$ and $f[a]$ is iso.

Let A, B be categories and let F_1, F_2 be covariant functors from A to B . We say that F_1 and F_2 are naturally equivalent if and only if the conditions (Def. 4) are satisfied.

- (Def. 4)(i) F_1 is naturally transformable to F_2 ,
- (ii) F_2 is transformable to F_1 , and
 - (iii) there exists a natural transformation t from F_1 to F_2 such that for every object a of A holds $t[a]$ is iso.

Let us notice that the predicate F_1 and F_2 are naturally equivalent is reflexive and symmetric.

Let A, B be categories and let F_1, F_2 be covariant functors from A to B . Let us assume that F_1 and F_2 are naturally equivalent. A natural transformation from F_1 to F_2 is said to be a natural equivalence of F_1 and F_2 if:

- (Def. 5) For every object a of A holds $t[a]$ is iso.

In the sequel e is a natural equivalence of F_1 and F_2 , e_1 is a natural equivalence of F_2 and F_3 , and f is a natural equivalence of G_1 and G_2 .

Next we state several propositions:

- (33) Suppose F_1 and F_2 are naturally equivalent and F_2 and F_3 are naturally equivalent. Then F_1 and F_3 are naturally equivalent.
- (34) Suppose F_1 and F_2 are naturally equivalent and F_2 and F_3 are naturally equivalent. Then $e_1 \circ e$ is a natural equivalence of F_1 and F_3 .
- (35) Suppose F_1 and F_2 are naturally equivalent. Then $G_1 \cdot F_1$ and $G_1 \cdot F_2$ are naturally equivalent and $G_1 \cdot e$ is a natural equivalence of $G_1 \cdot F_1$ and $G_1 \cdot F_2$.
- (36) Suppose G_1 and G_2 are naturally equivalent. Then $G_1 \cdot F_1$ and $G_2 \cdot F_1$ are naturally equivalent and $f \cdot F_1$ is a natural equivalence of $G_1 \cdot F_1$ and $G_2 \cdot F_1$.
- (37) Suppose F_1 and F_2 are naturally equivalent and G_1 and G_2 are naturally equivalent. Then $G_1 \cdot F_1$ and $G_2 \cdot F_2$ are naturally equivalent and $f e$ is a natural equivalence of $G_1 \cdot F_1$ and $G_2 \cdot F_2$.

Let A, B be categories, let F_1, F_2 be covariant functors from A to B , and let e be a natural equivalence of F_1 and F_2 . Let us assume that F_1 and F_2 are naturally equivalent. The functor e^{-1} yields a natural equivalence of F_2 and F_1 and is defined as follows:

(Def. 6) For every object a of A holds $e^{-1}(a) = e[a]^{-1}$.

The following three propositions are true:

- (38) For every object o of A such that F_1 and F_2 are naturally equivalent holds $e^{-1}[o] = e[o]^{-1}$.
- (39) If F_1 and F_2 are naturally equivalent, then $e \circ e^{-1} = \text{id}_{(F_2)}$.
- (40) If F_1 and F_2 are naturally equivalent, then $e^{-1} \circ e = \text{id}_{(F_1)}$.

Let A, B be categories and let F be a covariant functor from A to B . Then id_F is a natural equivalence of F and F .

The following propositions are true:

- (41) If F_1 and F_2 are naturally equivalent, then $(e^{-1})^{-1} = e$.
- (42) Let k be a natural equivalence of F_1 and F_3 . Suppose $k = e_1 \circ e$ and F_1 and F_2 are naturally equivalent and F_2 and F_3 are naturally equivalent. Then $k^{-1} = e^{-1} \circ e_1^{-1}$.
- (43) $(\text{id}_{(F_1)})^{-1} = \text{id}_{(F_1)}$.

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