JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Functors for Alternative Categories

Andrzej Trybulec
Warsaw University
Białystok

Summary. An attempt to define the concept of a functor covering both cases (covariant and contravariant) resulted in a structure consisting of two fields: the object map and the morphism map, the first one mapping the Cartesian squares of the set of objects rather than the set of objects. We start with an auxiliary notion of *bifunction*, i.e. a function mapping the Cartesian square of a set A into the Cartesian square of a set B. A *bifunction* f is said to be *covariant* if there is a function g from A into B that f is the Cartesian square of g and f is *contravariant* if there is a function g such that $f(o_1,o_2) = \langle g(o_2),g(o_1)\rangle$. The term *transformation*, another auxiliary notion, might be misleading. It is not related to natural transformations. A transformation from a many sorted set A indexed by I into a many sorted set B indexed by J w.r.t. a function f from I into J is a (many sorted) function from A into $B \cdot f$. Eventually, the morphism map of a functor from C_1 into C_2 is a transformation from the arrows of the category C_1 into the composition of the object map of the functor and the arrows of C_2.

Several kinds of functor structures have been defined: one-to-one, faithful, onto, full and id-preserving. We were pressed to split property that the composition be preserved into two: comp-preserving (for covariant functors) and comp-reversing (for contravariant functors). We defined also some operation on functors, e.g. the composition, the inverse functor. In the last section it is defined what is meant that two categories are isomorphic (anti-isomorphic).

MML Identifier: FUNCTOR0.

WWW: http://mizar.org/JFM/Vol8/functor0.html

The articles [11], [6], [17], [18], [19], [12], [3], [5], [4], [2], [10], [1], [7], [13], [9], [14], [8], [15], and [16] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *ValOnPair* deals with a non empty set $\mathcal{A}$, a function $\mathcal{B}$, elements $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

$\mathcal{B}(\mathcal{C},\mathcal{D}) = \mathcal{F}(\mathcal{C},\mathcal{D})$

provided the following conditions are met:

- $\mathcal{B} = \{\langle\langle o, o'\rangle, \mathcal{F}(o,o')\rangle ; o$ ranges over elements of $\mathcal{A}, o'$ ranges over elements of $\mathcal{A} : \mathcal{P}[o,o']\}$, and
- $\mathcal{P}[\mathcal{C},\mathcal{D}]$.

One can prove the following propositions:

(1) For every set A holds $\emptyset$ is a function from A into $\emptyset$.

(3)[1] For every set I and for every many sorted set M indexed by I holds $M \cdot \mathrm{id}_I = M$.

[1] The proposition (2) has been removed.

Let f be an empty function. Note that $\curvearrowleft f$ is empty. Let g be a function. One can verify that $[: f, g :]$ is empty and $[: g, f :]$ is empty.

The following two propositions are true:

(4) For every set A and for every function f holds $f^\circ(\mathrm{id}_A) = (\curvearrowleft f)^\circ(\mathrm{id}_A)$.

(5) For all sets X, Y and for every function f from X into Y holds f is onto iff $[: f, f :]$ is onto.

Let f be a function yielding function. Observe that $\curvearrowleft f$ is function yielding.

One can prove the following propositions:

(6) For all sets A, B and for every set a holds $\curvearrowleft([: A, B :] \longmapsto a) = [: B, A :] \longmapsto a$.

(7) For all functions f, g such that f is one-to-one and g is one-to-one holds $[: f, g :]^{-1} = [: f^{-1}, g^{-1} :]$.

(8) For every function f such that $[: f, f :]$ is one-to-one holds f is one-to-one.

(9) For every function f such that f is one-to-one holds $\curvearrowleft f$ is one-to-one.

(10) For all functions f, g such that $\curvearrowleft[: f, g :]$ is one-to-one holds $[: g, f :]$ is one-to-one.

(11) For all functions f, g such that f is one-to-one and g is one-to-one holds $(\curvearrowleft[: f, g :])^{-1} = \curvearrowleft([: g, f :]^{-1})$.

(12) For all sets A, B and for every function f from A into B such that f is onto holds $\mathrm{id}_B \subseteq [: f, f :]^\circ(\mathrm{id}_A)$.

(13) For all function yielding functions F, G and for every function f holds $(G \circ F) \cdot f = (G \cdot f) \circ (F \cdot f)$.

Let A, B, C be sets and let f be a function from $[: A, B :]$ into C. Then $\curvearrowleft f$ is a function from $[: B, A :]$ into C.

We now state two propositions:

(14) For all sets A, B, C and for every function f from $[: A, B :]$ into C such that $\curvearrowleft f$ is onto holds f is onto.

(15) For every set A and for every non empty set B and for every function f from A into B holds $[: f, f :]^\circ(\mathrm{id}_A) \subseteq \mathrm{id}_B$.

2. Functions Between Cartesian Squares

Let A, B be sets. A bifunction from A into B is a function from $[: A, A :]$ into $[: B, B :]$.

Let A, B be sets and let f be a bifunction from A into B. We say that f is precovariant if and only if:

(Def. 2)[2] There exists a function g from A into B such that $f = [: g, g :]$.

We say that f is precontravariant if and only if:

(Def. 3) There exists a function g from A into B such that $f = \curvearrowleft[: g, g :]$.

One can prove the following proposition

(16) Let A be a set, B be a non empty set, b be an element of B, and f be a bifunction from A into B. If $f = [: A, A :] \longmapsto \langle b, b \rangle$, then f is precovariant and precontravariant.

Let A, B be sets. Observe that there exists a bifunction from A into B which is precovariant and precontravariant.

The following proposition is true

(17) Let A, B be non empty sets and f be a precovariant precontravariant bifunction from A into B. Then there exists an element b of B such that $f = [: A, A :] \longmapsto \langle b, b \rangle$.

[2] The definition (Def. 1) has been removed.

3. Unary Transformations

Let I_1, I_2 be sets, let f be a function from I_1 into I_2, let A be a many sorted set indexed by I_1, and let B be a many sorted set indexed by I_2. A many sorted set indexed by I_1 is said to be a f-transformation from A to B if:

(Def. 4)(i) There exists a non empty set I'_2 and there exists a many sorted set B' indexed by I'_2 and there exists a function f' from I_1 into I'_2 such that $f = f'$ and $B = B'$ and it is a many sorted function from A into $B' \cdot f'$ if $I_2 \neq \emptyset$,

(ii) it $= \mathbf{0}_{(I_1)}$, otherwise.

Let I_1 be a set, let I_2 be a non empty set, let f be a function from I_1 into I_2, let A be a many sorted set indexed by I_1, and let B be a many sorted set indexed by I_2. Let us note that the f-transformation from A to B can be characterized by the following (equivalent) condition:

(Def. 5) It is a many sorted function from A into $B \cdot f$.

Let I_1, I_2 be sets, let f be a function from I_1 into I_2, let A be a many sorted set indexed by I_1, and let B be a many sorted set indexed by I_2. One can check that every f-transformation from A to B is function yielding.

Next we state the proposition

(18) Let I_1 be a set, I_2, I_3 be non empty sets, f be a function from I_1 into I_2, g be a function from I_2 into I_3, B be a many sorted set indexed by I_2, C be a many sorted set indexed by I_3, and G be a g-transformation from B to C. Then $G \cdot f$ is a $g \cdot f$-transformation from $B \cdot f$ to C.

Let I_1 be a set, let I_2 be a non empty set, let f be a function from I_1 into I_2, let A be a many sorted set indexed by $[\!: I_1, I_1 :\!]$, let B be a many sorted set indexed by $[\!: I_2, I_2 :\!]$, and let F be a $[\!: f, f :\!]$-transformation from A to B. Then $\curvearrowleft F$ is a $[\!: f, f :\!]$-transformation from $\curvearrowleft A$ to $\curvearrowleft B$.

The following propositions are true:

(19) Let I_1, I_2 be non empty sets, A be a many sorted set indexed by I_1, B be a many sorted set indexed by I_2, and o be an element of I_2. Suppose $B(o) \neq \emptyset$. Let m be an element of $B(o)$ and f be a function from I_1 into I_2. Suppose $f = I_1 \longmapsto o$. Then $\{\langle o', A(o') \longmapsto m \rangle : o'$ ranges over elements of $I_1\}$ is a f-transformation from A to B.

(20) Let I_1 be a set, I_2, I_3 be non empty sets, f be a function from I_1 into I_2, g be a function from I_2 into I_3, A be a many sorted set indexed by I_1, B be a many sorted set indexed by I_2, C be a many sorted set indexed by I_3, F be a f-transformation from A to B, and G be a $g \cdot f$-transformation from $B \cdot f$ to C. Suppose that for every set i_1 such that $i_1 \in I_1$ and $(B \cdot f)(i_1) = \emptyset$ holds $A(i_1) = \emptyset$ or $(C \cdot (g \cdot f))(i_1) = \emptyset$. Then $G \circ (F$ **qua** function yielding function) is a $g \cdot f$-transformation from A to C.

4. Functors

Let C_1, C_2 be 1-sorted structures. We consider bimap structures from C_1 into C_2 as systems

⟨ an object map ⟩,

where the object map is a bifunction from the carrier of C_1 into the carrier of C_2.

Let C_1, C_2 be non empty graphs, let F be a bimap structure from C_1 into C_2, and let o be an object of C_1. The functor $F(o)$ yielding an object of C_2 is defined as follows:

(Def. 6) $F(o) = (\text{the object map of } F)(o, o)_{\mathbf{1}}$.

Let A, B be 1-sorted structures and let F be a bimap structure from A into B. We say that F is one-to-one if and only if:

(Def. 7) The object map of F is one-to-one.

We say that F is onto if and only if:

(Def. 8) The object map of F is onto.

We say that F is reflexive if and only if:

(Def. 9) (The object map of $F)^{\circ}(\mathrm{id}_{\text{the carrier of } A}) \subseteq \mathrm{id}_{\text{the carrier of } B}$.

We say that F is coreflexive if and only if:

(Def. 10) $\mathrm{id}_{\text{the carrier of } B} \subseteq$ (the object map of $F)^{\circ}(\mathrm{id}_{\text{the carrier of } A})$.

Let A, B be non empty graphs and let F be a bimap structure from A into B. Let us observe that F is reflexive if and only if:

(Def. 11) For every object o of A holds (the object map of $F)(o,o) = \langle F(o), F(o) \rangle$.

Next we state the proposition

(21) Let A, B be reflexive non empty graphs and F be a bimap structure from A into B. Suppose F is coreflexive. Let o be an object of B. Then there exists an object o' of A such that $F(o') = o$.

Let C_1, C_2 be non empty graphs and let F be a bimap structure from C_1 into C_2. We say that F is feasible if and only if:

(Def. 12) For all objects o_1, o_2 of C_1 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds (the arrows of C_2)((the object map of $F)(o_1, o_2)) \neq \emptyset$.

Let C_1, C_2 be graphs. We introduce functor structures from C_1 to C_2 which are extensions of bimap structure from C_1 into C_2 and are systems
⟨ an object map, a morphism map ⟩,
where the object map is a bifunction from the carrier of C_1 into the carrier of C_2 and the morphism map is a the object map-transformation from the arrows of C_1 to the arrows of C_2.

Let C_1, C_2 be 1-sorted structures and let I_4 be a bimap structure from C_1 into C_2. We say that I_4 is precovariant if and only if:

(Def. 13) The object map of I_4 is precovariant.

We say that I_4 is precontravariant if and only if:

(Def. 14) The object map of I_4 is precontravariant.

Let C_1, C_2 be graphs. One can check that there exists a functor structure from C_1 to C_2 which is precovariant and there exists a functor structure from C_1 to C_2 which is precontravariant.

Let C_1, C_2 be graphs, let F be a functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. The functor $\text{Morph-Map}_F(o_1,o_2)$ is defined as follows:

(Def. 15) $\text{Morph-Map}_F(o_1,o_2) =$ (the morphism map of $F)(o_1, o_2)$.

Let C_1, C_2 be graphs, let F be a functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Observe that $\text{Morph-Map}_F(o_1,o_2)$ is relation-like and function-like.

Let C_1, C_2 be non empty graphs, let F be a precovariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Then $\text{Morph-Map}_F(o_1,o_2)$ is a function from $\langle o_1, o_2 \rangle$ into $\langle F(o_1), F(o_2) \rangle$.

Let C_1, C_2 be non empty graphs, let F be a precovariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Let us assume that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle F(o_1), F(o_2) \rangle \neq \emptyset$. Let m be a morphism from o_1 to o_2. The functor $F(m)$ yielding a morphism from $F(o_1)$ to $F(o_2)$ is defined as follows:

(Def. 16) $F(m) = (\text{Morph-Map}_F(o_1,o_2))(m)$.

Let C_1, C_2 be non empty graphs, let F be a precontravariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Then $\text{Morph-Map}_F(o_1,o_2)$ is a function from $\langle o_1, o_2 \rangle$ into $\langle F(o_2), F(o_1) \rangle$.

Let C_1, C_2 be non empty graphs, let F be a precontravariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Let us assume that $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle F(o_2), F(o_1) \rangle \neq \emptyset$. Let m be a morphism from o_1 to o_2. The functor $F(m)$ yields a morphism from $F(o_2)$ to $F(o_1)$ and is defined as follows:

(Def. 17) $F(m) = (\text{Morph-Map}_F(o_1,o_2))(m)$.

Let C_1, C_2 be non empty graphs and let o be an object of C_2. Let us assume that $\langle o,o\rangle \neq \emptyset$. Let m be a morphism from o to o. The functor $C_1 \longmapsto m$ yielding a strict functor structure from C_1 to C_2 is defined by the conditions (Def. 18).

(Def. 18)(i) The object map of $(C_1 \longmapsto m) = [\!:$the carrier of C_1, the carrier of $C_1 :\!] \longmapsto \langle o, o\rangle$, and

(ii) the morphism map of $(C_1 \longmapsto m) = \{\langle\langle o_1, o_2\rangle, (\langle o_1,o_2\rangle) \longmapsto m\rangle : o_1$ ranges over objects of C_1, o_2 ranges over objects of $C_1\}$.

Next we state the proposition

(22) Let C_1, C_2 be non empty graphs and o_2 be an object of C_2. Suppose $\langle o_2,o_2\rangle \neq \emptyset$. Let m be a morphism from o_2 to o_2 and o_1 be an object of C_1. Then $(C_1 \longmapsto m)(o_1) = o_2$.

Let C_1 be a non empty graph, let C_2 be a non empty reflexive graph, let o be an object of C_2, and let m be a morphism from o to o. Note that $C_1 \longmapsto m$ is precovariant, precontravariant, and feasible.

Let C_1 be a non empty graph and let C_2 be a non empty reflexive graph. One can verify that there exists a functor structure from C_1 to C_2 which is feasible, precovariant, and precontravariant.

Next we state the proposition

(23) Let C_1, C_2 be non empty graphs, F be a precovariant functor structure from C_1 to C_2, and o_1, o_2 be objects of C_1. Then (the object map of $F)(o_1, o_2) = \langle F(o_1), F(o_2)\rangle$.

Let C_1, C_2 be non empty graphs and let F be a precovariant functor structure from C_1 to C_2. Let us observe that F is feasible if and only if:

(Def. 19) For all objects o_1, o_2 of C_1 such that $\langle o_1,o_2\rangle \neq \emptyset$ holds $\langle F(o_1),F(o_2)\rangle \neq \emptyset$.

One can prove the following proposition

(24) Let C_1, C_2 be non empty graphs, F be a precontravariant functor structure from C_1 to C_2, and o_1, o_2 be objects of C_1. Then (the object map of $F)(o_1, o_2) = \langle F(o_2), F(o_1)\rangle$.

Let C_1, C_2 be non empty graphs and let F be a precontravariant functor structure from C_1 to C_2. Let us observe that F is feasible if and only if:

(Def. 20) For all objects o_1, o_2 of C_1 such that $\langle o_1,o_2\rangle \neq \emptyset$ holds $\langle F(o_2),F(o_1)\rangle \neq \emptyset$.

Let C_1, C_2 be graphs and let F be a functor structure from C_1 to C_2. Note that the morphism map of F is function yielding.

One can verify that there exists a category structure which is non empty and reflexive.

Let C_1, C_2 be non empty category structures with units and let F be a functor structure from C_1 to C_2. We say that F is id-preserving if and only if:

(Def. 21) For every object o of C_1 holds $(\text{Morph-Map}_F(o,o))(\text{id}_o) = \text{id}_{F(o)}$.

We now state the proposition

(25) Let C_1, C_2 be non empty graphs and o_2 be an object of C_2. Suppose $\langle o_2,o_2\rangle \neq \emptyset$. Let m be a morphism from o_2 to o_2, o, o' be objects of C_1, and f be a morphism from o to o'. If $\langle o,o'\rangle \neq \emptyset$, then $(\text{Morph-Map}_{C_1 \longmapsto m}(o,o'))(f) = m$.

One can verify that every non empty category structure which has units is also reflexive.

Let C_1, C_2 be non empty category structures with units and let o_2 be an object of C_2. One can check that $C_1 \longmapsto \text{id}_{(o_2)}$ is id-preserving.

Let C_1 be a non empty graph, let C_2 be a non empty reflexive graph, let o_2 be an object of C_2, and let m be a morphism from o_2 to o_2. Observe that $C_1 \longmapsto m$ is reflexive.

Let C_1 be a non empty graph and let C_2 be a non empty reflexive graph. One can check that there exists a functor structure from C_1 to C_2 which is feasible and reflexive.

Let C_1, C_2 be non empty category structures with units. Observe that there exists a functor structure from C_1 to C_2 which is id-preserving, feasible, reflexive, and strict.

Let C_1, C_2 be non empty category structures and let F be a functor structure from C_1 to C_2. We say that F is comp-preserving if and only if the condition (Def. 22) is satisfied.

(Def. 22) Let o_1, o_2, o_3 be objects of C_1. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and g be a morphism from o_2 to o_3. Then there exists a morphism f' from $F(o_1)$ to $F(o_2)$ and there exists a morphism g' from $F(o_2)$ to $F(o_3)$ such that $f' = (\text{Morph-Map}_F(o_1, o_2))(f)$ and $g' = (\text{Morph-Map}_F(o_2, o_3))(g)$ and $(\text{Morph-Map}_F(o_1, o_3))(g \cdot f) = g' \cdot f'$.

Let C_1, C_2 be non empty category structures and let F be a functor structure from C_1 to C_2. We say that F is comp-reversing if and only if the condition (Def. 23) is satisfied.

(Def. 23) Let o_1, o_2, o_3 be objects of C_1. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and g be a morphism from o_2 to o_3. Then there exists a morphism f' from $F(o_2)$ to $F(o_1)$ and there exists a morphism g' from $F(o_3)$ to $F(o_2)$ such that $f' = (\text{Morph-Map}_F(o_1, o_2))(f)$ and $g' = (\text{Morph-Map}_F(o_2, o_3))(g)$ and $(\text{Morph-Map}_F(o_1, o_3))(g \cdot f) = f' \cdot g'$.

Let C_1 be a non empty transitive category structure, let C_2 be a non empty reflexive category structure, and let F be a precovariant feasible functor structure from C_1 to C_2. Let us observe that F is comp-preserving if and only if the condition (Def. 24) is satisfied.

(Def. 24) Let o_1, o_2, o_3 be objects of C_1. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and g be a morphism from o_2 to o_3. Then $F(g \cdot f) = F(g) \cdot F(f)$.

Let C_1 be a non empty transitive category structure, let C_2 be a non empty reflexive category structure, and let F be a precontravariant feasible functor structure from C_1 to C_2. Let us observe that F is comp-reversing if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let o_1, o_2, o_3 be objects of C_1. Suppose $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle o_2, o_3\rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and g be a morphism from o_2 to o_3. Then $F(g \cdot f) = F(f) \cdot F(g)$.

We now state two propositions:

(26) Let C_1 be a non empty graph, C_2 be a non empty reflexive graph, o_2 be an object of C_2, m be a morphism from o_2 to o_2, and F be a precovariant feasible functor structure from C_1 to C_2. Suppose $F = C_1 \longmapsto m$. Let o, o' be objects of C_1 and f be a morphism from o to o'. If $\langle o, o'\rangle \neq \emptyset$, then $F(f) = m$.

(27) Let C_1 be a non empty graph, C_2 be a non empty reflexive graph, o_2 be an object of C_2, m be a morphism from o_2 to o_2, o, o' be objects of C_1, and f be a morphism from o to o'. If $\langle o, o'\rangle \neq \emptyset$, then $(C_1 \longmapsto m)(f) = m$.

Let C_1 be a non empty transitive category structure, let C_2 be a non empty category structure with units, and let o be an object of C_2. Note that $C_1 \longmapsto \mathrm{id}_o$ is comp-preserving and comp-reversing.

Let C_1 be a transitive non empty category structure with units and let C_2 be a non empty category structure with units. A functor structure from C_1 to C_2 is said to be a functor from C_1 to C_2 if:

(Def. 26) It is feasible and id-preserving but it is precovariant and comp-preserving or it is precontravariant and comp-reversing.

Let C_1 be a transitive non empty category structure with units, let C_2 be a non empty category structure with units, and let F be a functor from C_1 to C_2. We say that F is covariant if and only if:

(Def. 27) F is precovariant and comp-preserving.

We say that F is contravariant if and only if:

(Def. 28) F is precontravariant and comp-reversing.

Let A be a category structure and let B be a substructure of A. The functor $\overset{B}{\hookrightarrow}$ yielding a strict functor structure from B to A is defined by the conditions (Def. 29).

(Def. 29)(i) The object map of $(\overset{B}{\hookrightarrow}) = \mathrm{id}_{[:\text{the carrier of } B, \text{the carrier of } B:]}$, and

(ii) the morphism map of $(\overset{B}{\hookrightarrow}) = \mathrm{id}_{\text{the arrows of } B}$.

Let A be a graph. The functor id_A yielding a strict functor structure from A to A is defined by the conditions (Def. 30).

(Def. 30)(i) The object map of $\mathrm{id}_A = \mathrm{id}_{[:\text{the carrier of } A, \text{ the carrier of } A:]}$, and

(ii) the morphism map of $\mathrm{id}_A = \mathrm{id}_{\text{the arrows of } A}$.

Let A be a category structure and let B be a substructure of A. One can check that $\overset{B}{\hookrightarrow}$ is precovariant.

We now state two propositions:

(28) Let A be a non empty category structure, B be a non empty substructure of A, and o be an object of B. Then $(\overset{B}{\hookrightarrow})(o) = o$.

(29) Let A be a non empty category structure, B be a non empty substructure of A, and o_1, o_2 be objects of B. Then $\langle o_1, o_2 \rangle \subseteq \langle (\overset{B}{\hookrightarrow})(o_1), (\overset{B}{\hookrightarrow})(o_2) \rangle$.

Let A be a non empty category structure and let B be a non empty substructure of A. One can verify that $\overset{B}{\hookrightarrow}$ is feasible.

Let A, B be graphs and let F be a functor structure from A to B. We say that F is faithful if and only if:

(Def. 31) The morphism map of F is "1-1".

Let A, B be graphs and let F be a functor structure from A to B. We say that F is full if and only if the condition (Def. 32) is satisfied.

(Def. 32) There exists a many sorted set B' indexed by $[:$the carrier of A, the carrier of $A:]$ and there exists a many sorted function f from the arrows of A into B' such that $B' =$ (the arrows of $B) \cdot$ (the object map of F) and $f =$ the morphism map of F and f is onto.

Let A be a graph, let B be a non empty graph, and let F be a functor structure from A to B. Let us observe that F is full if and only if the condition (Def. 33) is satisfied.

(Def. 33) There exists a many sorted function f from the arrows of A into (the arrows of $B) \cdot$ (the object map of F) such that $f =$ the morphism map of F and f is onto.

Let A, B be graphs and let F be a functor structure from A to B. We say that F is injective if and only if:

(Def. 34) F is one-to-one and faithful.

We say that F is surjective if and only if:

(Def. 35) F is full and onto.

Let A, B be graphs and let F be a functor structure from A to B. We say that F is bijective if and only if:

(Def. 36) F is injective and surjective.

Let A, B be transitive non empty category structures with units. Note that there exists a functor from A to B which is strict, covariant, contravariant, and feasible.

Next we state two propositions:

(30) For every non empty graph A and for every object o of A holds $\mathrm{id}_A(o) = o$.

(31) Let A be a non empty graph and o_1, o_2 be objects of A. If $\langle o_1, o_2 \rangle \neq \emptyset$, then for every morphism m from o_1 to o_2 holds $(\text{Morph-Map}_{\mathrm{id}_A}(o_1, o_2))(m) = m$.

Let A be a non empty graph. One can check that id_A is feasible and precovariant.

Let A be a non empty graph. One can check that there exists a functor structure from A to A which is precovariant and feasible.

We now state the proposition

(32) Let A be a non empty graph and o_1, o_2 be objects of A. Suppose $\langle o_1, o_2 \rangle \neq \emptyset$. Let F be a precovariant feasible functor structure from A to A. If $F = \mathrm{id}_A$, then for every morphism m from o_1 to o_2 holds $F(m) = m$.

Let A be a transitive non empty category structure with units. Observe that id_A is id-preserving and comp-preserving.

Let A be a transitive non empty category structure with units. Then id_A is a strict covariant functor from A to A.

Let A be a graph. Note that id_A is bijective.

5. The Composition of Functors

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3. The functor $G \cdot F$ yields a strict functor structure from C_1 to C_3 and is defined by the conditions (Def. 37).

(Def. 37)(i) The object map of $G \cdot F =$ (the object map of G) $\cdot$ (the object map of F), and

(ii) the morphism map of $G \cdot F =$ ((the morphism map of G) $\cdot$ (the object map of F)) $\circ$ the morphism map of F.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precovariant feasible functor structure from C_1 to C_2, and let G be a precovariant functor structure from C_2 to C_3. Note that $G \cdot F$ is precovariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precontravariant feasible functor structure from C_1 to C_2, and let G be a precovariant functor structure from C_2 to C_3. Observe that $G \cdot F$ is precontravariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precovariant feasible functor structure from C_1 to C_2, and let G be a precontravariant functor structure from C_2 to C_3. Observe that $G \cdot F$ is precontravariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precontravariant feasible functor structure from C_1 to C_2, and let G be a precontravariant functor structure from C_2 to C_3. Note that $G \cdot F$ is precovariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a feasible functor structure from C_1 to C_2, and let G be a feasible functor structure from C_2 to C_3. One can check that $G \cdot F$ is feasible.

We now state three propositions:

(33) Let C_1 be a non empty graph, C_2, C_3, C_4 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, G be a feasible functor structure from C_2 to C_3, and H be a functor structure from C_3 to C_4. Then $(H \cdot G) \cdot F = H \cdot (G \cdot F)$.

(34) Let C_1 be a non empty category structure, C_2, C_3 be non empty reflexive category structures, F be a feasible reflexive functor structure from C_1 to C_2, G be a functor structure from C_2 to C_3, and o be an object of C_1. Then $(G \cdot F)(o) = G(F(o))$.

(35) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible reflexive functor structure from C_1 to C_2, G be a functor structure from C_2 to C_3, and o be an object of C_1. Then $\text{Morph-Map}_{G \cdot F}(o, o) = \text{Morph-Map}_G(F(o), F(o)) \cdot \text{Morph-Map}_F(o, o)$.

Let C_1, C_2, C_3 be non empty category structures with units, let F be an id-preserving feasible reflexive functor structure from C_1 to C_2, and let G be an id-preserving functor structure from C_2 to C_3. Observe that $G \cdot F$ is id-preserving.

Let A, C be non empty reflexive category structures, let B be a non empty substructure of A, and let F be a functor structure from A to C. The functor $F \restriction B$ yielding a functor structure from B to C is defined by:

(Def. 38) $F \restriction B = F \cdot (\overset{B}{\hookrightarrow})$.

6. THE INVERSE FUNCTOR

Let A, B be non empty graphs and let F be a functor structure from A to B. Let us assume that F is bijective. The functor F^{-1} yields a strict functor structure from B to A and is defined by the conditions (Def. 39).

(Def. 39)(i) The object map of F^{-1} = (the object map of $F)^{-1}$, and

(ii) there exists a many sorted function f from the arrows of A into (the arrows of $B) \cdot$ (the object map of F) such that f = the morphism map of F and the morphism map of $F^{-1} = f^{-1} \cdot$ (the object map of $F)^{-1}$.

Next we state several propositions:

(36) Let A, B be transitive non empty category structures with units and F be a feasible functor structure from A to B. If F is bijective, then F^{-1} is bijective and feasible.

(37) Let A, B be transitive non empty category structures with units and F be a feasible reflexive functor structure from A to B. If F is bijective and coreflexive, then F^{-1} is reflexive.

(38) Let A, B be transitive non empty category structures with units and F be a feasible reflexive id-preserving functor structure from A to B. If F is bijective and coreflexive, then F^{-1} is id-preserving.

(39) Let A, B be transitive non empty category structures with units and F be a feasible functor structure from A to B. If F is bijective and precovariant, then F^{-1} is precovariant.

(40) Let A, B be transitive non empty category structures with units and F be a feasible functor structure from A to B. If F is bijective and precontravariant, then F^{-1} is precontravariant.

(41) Let A, B be transitive non empty category structures with units and F be a feasible reflexive functor structure from A to B. Suppose F is bijective, coreflexive, and precovariant. Let o_1, o_2 be objects of B and m be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(\text{Morph-Map}_F(F^{-1}(o_1), F^{-1}(o_2)))((\text{Morph-Map}_{F^{-1}}(o_1, o_2))(m)) = m$.

(42) Let A, B be transitive non empty category structures with units and F be a feasible reflexive functor structure from A to B. Suppose F is bijective, coreflexive, and precontravariant. Let o_1, o_2 be objects of B and m be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(\text{Morph-Map}_F(F^{-1}(o_2), F^{-1}(o_1)))((\text{Morph-Map}_{F^{-1}}(o_1, o_2))(m)) = m$.

(43) Let A, B be transitive non empty category structures with units and F be a feasible reflexive functor structure from A to B. Suppose F is bijective, comp-preserving, precovariant, and coreflexive. Then F^{-1} is comp-preserving.

(44) Let A, B be transitive non empty category structures with units and F be a feasible reflexive functor structure from A to B. Suppose F is bijective, comp-reversing, precontravariant, and coreflexive. Then F^{-1} is comp-reversing.

Let C_1 be a 1-sorted structure and let C_2 be a non empty 1-sorted structure. Observe that every bimap structure from C_1 into C_2 which is precovariant is also reflexive.

Let C_1 be a 1-sorted structure and let C_2 be a non empty 1-sorted structure. One can check that every bimap structure from C_1 into C_2 which is precontravariant is also reflexive.

The following two propositions are true:

(45) Let C_1, C_2 be 1-sorted structures and M be a bimap structure from C_1 into C_2. If M is precovariant and onto, then M is coreflexive.

(46) Let C_1, C_2 be 1-sorted structures and M be a bimap structure from C_1 into C_2. If M is precontravariant and onto, then M is coreflexive.

Let C_1 be a transitive non empty category structure with units and let C_2 be a non empty category structure with units. Note that every functor from C_1 to C_2 which is covariant is also reflexive.

Let C_1 be a transitive non empty category structure with units and let C_2 be a non empty category structure with units. One can check that every functor from C_1 to C_2 which is contravariant is also reflexive.

Next we state four propositions:

(47) Let C_1 be a transitive non empty category structure with units, C_2 be a non empty category structure with units, and F be a functor from C_1 to C_2. If F is covariant and onto, then F is coreflexive.

(48) Let C_1 be a transitive non empty category structure with units, C_2 be a non empty category structure with units, and F be a functor from C_1 to C_2. If F is contravariant and onto, then F is coreflexive.

(49) Let A, B be transitive non empty category structures with units and F be a covariant functor from A to B. Suppose F is bijective. Then there exists a functor G from B to A such that $G = F^{-1}$ and G is bijective and covariant.

(50) Let A, B be transitive non empty category structures with units and F be a contravariant functor from A to B. Suppose F is bijective. Then there exists a functor G from B to A such that $G = F^{-1}$ and G is bijective and contravariant.

Let A, B be transitive non empty category structures with units. We say that A and B are isomorphic if and only if:

(Def. 40) There exists a functor from A to B which is bijective and covariant.

Let us notice that the predicate A and B are isomorphic is reflexive and symmetric. We say that A, B are anti-isomorphic if and only if:

(Def. 41) There exists a functor from A to B which is bijective and contravariant.

Let us note that the predicate A, B are anti-isomorphic is symmetric.

REFERENCES

[1] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_3.html`.

[2] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[8] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_3.html`.

[9] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pralg_1.html`.

[10] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funcop_1.html`.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[12] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[13] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[14] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[15] Andrzej Trybulec. Categories without uniqueness of cod and dom. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/altcat_1.html.

[16] Andrzej Trybulec. Examples of category structures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/altcat_2.html.

[17] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[18] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[19] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received April 24, 1996

Published January 2, 2004