JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
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Miscellaneous Facts about Functions

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MML Identifier: FUNCT_7.

WWW: http://mizar.org/JFM/Vol8/funct_7.html

The articles [21], [10], [25], [23], [2], [17], [22], [20], [1], [16], [26], [7], [27], [6], [5], [15], [11], [14], [9], [19], [8], [12], [13], [3], [24], [18], and [4] provide the notation and terminology for this paper.

1. Preliminaries

We follow the rules: a, x, A, B denote sets and m, n denote natural numbers.

The following propositions are true:

(1) For every function f and for every set X such that $\operatorname{rng} f \subseteq X$ holds $\operatorname{id}_X \cdot f = f$.

(2) Let X be a set, Y be a non empty set, and f be a function from X into Y. Suppose f is one-to-one. Let B be a subset of X and C be a subset of Y. If $C \subseteq f^\circ B$, then $f^{-1}(C) \subseteq B$.

(3) Let X, Y be non empty sets and f be a function from X into Y. Suppose f is one-to-one. Let x be an element of X and A be a subset of X. If $f(x) \in f^\circ A$, then $x \in A$.

(4) Let X, Y be non empty sets and f be a function from X into Y. Suppose f is one-to-one. Let x be an element of X, A be a subset of X, and B be a subset of Y. If $f(x) \in f^\circ A \setminus B$, then $x \in A \setminus f^{-1}(B)$.

(5) Let X, Y be non empty sets and f be a function from X into Y. Suppose f is one-to-one. Let y be an element of Y, A be a subset of X, and B be a subset of Y. If $y \in f^\circ A \setminus B$, then $f^{-1}(y) \in A \setminus f^{-1}(B)$.

(6) For every function f and for every set a such that $a \in \operatorname{dom} f$ holds $f \restriction \{a\} = a \dot{\longmapsto} f(a)$.

Let x, y be sets. Observe that $x \dot{\longmapsto} y$ is non empty.

Let x, y, a, b be sets. Observe that $[x \longmapsto a, y \longmapsto b]$ is non empty.

One can prove the following propositions:

(7) For every set I and for every many sorted set M indexed by I and for every set i such that $i \in I$ holds $i \dot{\longmapsto} M(i) = M \restriction \{i\}$.

(8) Let I, J be sets, M be a many sorted set indexed by $[: I, J :]$, and i, j be sets. If $i \in I$ and $j \in J$, then $[\langle i, j \rangle \mapsto M(i, j)] = M \restriction [: \{i\}, \{j\} :]$.

(10)[1] For all functions f, g, h such that $\operatorname{rng} g \subseteq \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} f$ holds $f \cdot (g {+\cdot} h) = f \cdot g {+\cdot} f \cdot h$.

(11) For all functions f, g, h holds $(g {+\cdot} h) \cdot f = g \cdot f {+\cdot} h \cdot f$.

(12) For all functions f, g, h such that $\operatorname{rng} f$ misses $\operatorname{dom} g$ holds $(h {+\cdot} g) \cdot f = h \cdot f$.

(13) For all sets A, B and for every set y such that A meets $\operatorname{rng}(\mathrm{id}_B {+\cdot} (A \longmapsto y))$ holds $y \in A$.

(14) For all sets x, y and for every set A such that $x \neq y$ holds $x \notin \operatorname{rng}(\mathrm{id}_A {+\cdot} (x \mapsto\!\!\!\!\!\!\cdot\;\; y))$.

(15) For every set X and for every set a and for every function f such that $\operatorname{dom} f = X \cup \{a\}$ holds $f = f{\restriction}X {+\cdot} (a \mapsto\!\!\!\!\!\!\cdot\;\; f(a))$.

(16) For every function f and for all sets X, y, z holds $f {+\cdot} (X \longmapsto y) {+\cdot} (X \longmapsto z) = f {+\cdot} (X \longmapsto z)$.

(17) If $0 < m$ and $m \leq n$, then $\mathbb{Z}_m \subseteq \mathbb{Z}_n$.

(18) $\mathbb{Z} \neq \mathbb{Z}^*$.

(19) $\emptyset^* = \{\emptyset\}$.

(20) $\langle x \rangle \in A^*$ iff $x \in A$.

(21) $A \subseteq B$ iff $A^* \subseteq B^*$.

(22) For every subset A of $\mathbb{N}$ such that for all n, m such that $n \in A$ and $m < n$ holds $m \in A$ holds A is a cardinal number.

(23) Let A be a finite set and X be a non empty family of subsets of A. Then there exists an element C of X such that for every element B of X such that $B \subseteq C$ holds $B = C$.

(24) Let p, q be finite sequences. Suppose $\operatorname{len} p = \operatorname{len} q + 1$. Let i be a natural number. Then $i \in \operatorname{dom} q$ if and only if the following conditions are satisfied:

(i) $i \in \operatorname{dom} p$, and

(ii) $i + 1 \in \operatorname{dom} p$.

Let us note that there exists a finite sequence which is function yielding, non empty, and non-empty.

Observe that $\emptyset$ is function yielding. Let f be a function. One can verify that $\langle f \rangle$ is function yielding. Let g be a function. One can verify that $\langle f, g \rangle$ is function yielding. Let h be a function. Note that $\langle f, g, h \rangle$ is function yielding.

Let n be a natural number and let f be a function. Note that $n \mapsto f$ is function yielding.

Let p be a finite sequence and let q be a non empty finite sequence. Observe that $p \frown q$ is non empty and $q \frown p$ is non empty.

Let p, q be function yielding finite sequences. Observe that $p \frown q$ is function yielding.

One can prove the following proposition

(25) Let p, q be finite sequences. Suppose $p \frown q$ is function yielding. Then p is function yielding and q is function yielding.

[1] The proposition (9) has been removed.

2. Some useful schemes

In this article we present several logical schemes. The scheme *Kappa2D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $f(\langle x, y\rangle) = \mathcal{F}(x,y)$

provided the following requirement is met:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{F}(x,y) \in \mathcal{C}$.

The scheme *FinMono* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and states that:

$\{\mathcal{F}(d); d$ ranges over elements of $\mathcal{B}: \mathcal{G}(d) \in \mathcal{A}\}$ is finite

provided the parameters have the following properties:

- $\mathcal{A}$ is finite, and
- For all elements d_1, d_2 of $\mathcal{B}$ such that $\mathcal{G}(d_1) = \mathcal{G}(d_2)$ holds $d_1 = d_2$.

The scheme *CardMono* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A} \approx \{d; d$ ranges over elements of $\mathcal{B}: \mathcal{F}(d) \in \mathcal{A}\}$

provided the following conditions are met:

- For every set x such that $x \in \mathcal{A}$ there exists an element d of $\mathcal{B}$ such that $x = \mathcal{F}(d)$, and
- For all elements d_1, d_2 of $\mathcal{B}$ such that $\mathcal{F}(d_1) = \mathcal{F}(d_2)$ holds $d_1 = d_2$.

The scheme *CardMono'* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A} \approx \{\mathcal{F}(d); d$ ranges over elements of $\mathcal{B}: d \in \mathcal{A}\}$

provided the following requirements are met:

- $\mathcal{A} \subseteq \mathcal{B}$, and
- For all elements d_1, d_2 of $\mathcal{B}$ such that $\mathcal{F}(d_1) = \mathcal{F}(d_2)$ holds $d_1 = d_2$.

The scheme *FuncSeqInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every function yielding finite sequence p holds $\mathcal{P}[p]$

provided the parameters meet the following requirements:

- $\mathcal{P}[\emptyset]$, and
- For every function yielding finite sequence p such that $\mathcal{P}[p]$ and for every function f holds $\mathcal{P}[p \frown \langle f\rangle]$.

3. Some auxiliary concepts

Let x, y be sets. Let us assume that $x \in y$. The functor $x(\in y)$ yielding an element of y is defined as follows:

(Def. 1) $x(\in y) = x$.

One can prove the following proposition

(26) If $x \in A \cap B$, then $x(\in A) = x(\in B)$.

Let f, g be functions and let A be a set. We say that f and g are equal outside A if and only if:

(Def. 2) $f\restriction(\operatorname{dom} f \setminus A) = g\restriction(\operatorname{dom} g \setminus A)$.

We now state several propositions:

(27) For every function f and for every set A holds f and f are equal outside A.

(28) Let f, g be functions and A be a set. Suppose f and g are equal outside A. Then g and f are equal outside A.

(29) Let f, g, h be functions and A be a set. Suppose f and g are equal outside A and g and h are equal outside A. Then f and h are equal outside A.

(30) For all functions f, g and for every set A such that f and g are equal outside A holds $\operatorname{dom} f \setminus A = \operatorname{dom} g \setminus A$.

(31) For all functions f, g and for every set A such that $\operatorname{dom} g \subseteq A$ holds f and $f {+\cdot} g$ are equal outside A.

Let f be a function and let i, x be sets. The functor $f {+\cdot} (i,x)$ yielding a function is defined by:

(Def. 3) $$f {+\cdot} (i,x) = \begin{cases} f {+\cdot} (i \mapsto x), & \text{if } i \in \operatorname{dom} f, \\ f, & \text{otherwise.} \end{cases}$$

We now state several propositions:

(32) For every function f and for all sets d, i holds $\operatorname{dom}(f {+\cdot} (i,d)) = \operatorname{dom} f$.

(33) For every function f and for all sets d, i such that $i \in \operatorname{dom} f$ holds $(f {+\cdot} (i,d))(i) = d$.

(34) For every function f and for all sets d, i, j such that $i \neq j$ holds $(f {+\cdot} (i,d))(j) = f(j)$.

(35) For every function f and for all sets d, e, i, j such that $i \neq j$ holds $f {+\cdot} (i,d) {+\cdot} (j,e) = f {+\cdot} (j,e) {+\cdot} (i,d)$.

(36) For every function f and for all sets d, e, i holds $f {+\cdot} (i,d) {+\cdot} (i,e) = f {+\cdot} (i,e)$.

(37) For every function f and for every set i holds $f {+\cdot} (i, f(i)) = f$.

Let f be a finite sequence, let i be a natural number, and let x be a set. Note that $f {+\cdot} (i,x)$ is finite sequence-like.

Let D be a set, let f be a finite sequence of elements of D, let i be a natural number, and let d be an element of D. Then $f {+\cdot} (i,d)$ is a finite sequence of elements of D.

Next we state three propositions:

(38) Let D be a non empty set, f be a finite sequence of elements of D, d be an element of D, and i be a natural number. If $i \in \operatorname{dom} f$, then $(f {+\cdot} (i,d))_i = d$.

(39) Let D be a non empty set, f be a finite sequence of elements of D, d be an element of D, and i, j be natural numbers. If $i \neq j$ and $j \in \operatorname{dom} f$, then $(f {+\cdot} (i,d))_j = f_j$.

(40) Let D be a non empty set, f be a finite sequence of elements of D, d, e be elements of D, and i be a natural number. Then $f {+\cdot} (i, f_i) = f$.

4. On the Composition of a Finite Sequence of Functions

Let X be a set and let p be a function yielding finite sequence. The functor $\operatorname{compose}_X p$ yields a function and is defined by the condition (Def. 4).

(Def. 4) There exists a many sorted function f indexed by $\mathbb{N}$ such that

(i) $\operatorname{compose}_X p = f(\operatorname{len} p)$,

(ii) $f(0) = \operatorname{id}_X$, and

(iii) for every natural number i such that $i+1 \in \operatorname{dom} p$ and for all functions g, h such that $g = f(i)$ and $h = p(i+1)$ holds $f(i+1) = h \cdot g$.

Let p be a function yielding finite sequence and let x be a set. The functor $\operatorname{apply}(p,x)$ yields a finite sequence and is defined by the conditions (Def. 5).

(Def. 5)(i) $\operatorname{len} \operatorname{apply}(p,x) = \operatorname{len} p + 1$,

(ii) $(\operatorname{apply}(p,x))(1) = x$, and

(iii) for every natural number i and for every function f such that $i \in \operatorname{dom} p$ and $f = p(i)$ holds $(\operatorname{apply}(p,x))(i+1) = f((\operatorname{apply}(p,x))(i))$.

We adopt the following convention: X, Y, x denote sets, p, q denote function yielding finite sequences, and f, g, h denote functions.

Next we state a number of propositions:

(41) $\mathrm{compose}_X\,\emptyset = \mathrm{id}_X$.

(42) $\mathrm{apply}(\emptyset, x) = \langle x \rangle$.

(43) $\mathrm{compose}_X(p \frown \langle f \rangle) = f \cdot \mathrm{compose}_X\,p$.

(44) $\mathrm{apply}(p \frown \langle f \rangle, x) = (\mathrm{apply}(p,x)) \frown \langle f((\mathrm{apply}(p,x))(\mathrm{len}\,p+1)) \rangle$.

(45) $\mathrm{compose}_X(\langle f \rangle \frown p) = \mathrm{compose}_{f^\circ X}\,p \cdot (f \restriction X)$.

(46) $\mathrm{apply}(\langle f \rangle \frown p, x) = \langle x \rangle \frown \mathrm{apply}(p, f(x))$.

(47) $\mathrm{compose}_X\langle f \rangle = f \cdot \mathrm{id}_X$.

(48) If $\mathrm{dom}\,f \subseteq X$, then $\mathrm{compose}_X\langle f \rangle = f$.

(49) $\mathrm{apply}(\langle f \rangle, x) = \langle x, f(x) \rangle$.

(50) If $\mathrm{rng}\,\mathrm{compose}_X\,p \subseteq Y$, then $\mathrm{compose}_X(p \frown q) = \mathrm{compose}_Y\,q \cdot \mathrm{compose}_X\,p$.

(51) $(\mathrm{apply}(p \frown q, x))(\mathrm{len}(p \frown q)+1) = (\mathrm{apply}(q, (\mathrm{apply}(p,x))(\mathrm{len}\,p+1)))(\mathrm{len}\,q+1)$.

(52) $\mathrm{apply}(p \frown q, x) = (\mathrm{apply}(p,x)) \,{}^{\$}\!\frown \mathrm{apply}(q, (\mathrm{apply}(p,x))(\mathrm{len}\,p+1))$.

(53) $\mathrm{compose}_X\langle f, g \rangle = g \cdot f \cdot \mathrm{id}_X$.

(54) If $\mathrm{dom}\,f \subseteq X$ or $\mathrm{dom}(g \cdot f) \subseteq X$, then $\mathrm{compose}_X\langle f, g \rangle = g \cdot f$.

(55) $\mathrm{apply}(\langle f, g \rangle, x) = \langle x, f(x), g(f(x)) \rangle$.

(56) $\mathrm{compose}_X\langle f, g, h \rangle = h \cdot g \cdot f \cdot \mathrm{id}_X$.

(57) If $\mathrm{dom}\,f \subseteq X$ or $\mathrm{dom}(g \cdot f) \subseteq X$ or $\mathrm{dom}(h \cdot g \cdot f) \subseteq X$, then $\mathrm{compose}_X\langle f, g, h \rangle = h \cdot g \cdot f$.

(58) $\mathrm{apply}(\langle f, g, h \rangle, x) = \langle x \rangle \frown \langle f(x), g(f(x)), h(g(f(x))) \rangle$.

Let F be a finite sequence. The functor $\mathrm{firstdom}(F)$ is defined by:

(Def. 6)(i) $\mathrm{firstdom}(F)$ is empty if F is empty,

(ii) $\mathrm{firstdom}(F) = \pi_1(F(1))$, otherwise.

The functor $\mathrm{lastrng}(F)$ is defined as follows:

(Def. 7)(i) $\mathrm{lastrng}(F)$ is empty if F is empty,

(ii) $\mathrm{lastrng}(F) = \pi_2(F(\mathrm{len}\,F))$, otherwise.

The following three propositions are true:

(59) $\mathrm{firstdom}(\emptyset) = \emptyset$ and $\mathrm{lastrng}(\emptyset) = \emptyset$.

(60) For every finite sequence p holds $\mathrm{firstdom}(\langle f \rangle \frown p) = \mathrm{dom}\,f$ and $\mathrm{lastrng}(p \frown \langle f \rangle) = \mathrm{rng}\,f$.

(61) For every function yielding finite sequence p such that $p \neq \emptyset$ holds $\mathrm{rng}\,\mathrm{compose}_X\,p \subseteq \mathrm{lastrng}(p)$.

Let I_1 be a finite sequence. We say that I_1 is composable if and only if:

(Def. 8) There exists a finite sequence p such that $\mathrm{len}\,p = \mathrm{len}\,I_1 + 1$ and for every natural number i such that $i \in \mathrm{dom}\,I_1$ holds $I_1(i) \in p(i+1)^{p(i)}$.

One can prove the following proposition

(62) For all finite sequences p, q such that $p \frown q$ is composable holds p is composable and q is composable.

One can check that every finite sequence which is composable is also function yielding.
Let us observe that every finite sequence which is empty is also composable.
Let f be a function. Observe that $\langle f \rangle$ is composable.
Let us note that there exists a finite sequence which is composable, non empty, and non-empty.
A composable sequence is a composable finite sequence.
Next we state several propositions:

(63) For every composable sequence p such that $p \neq \emptyset$ holds $\mathrm{dom}\,\mathrm{compose}_X\, p = \mathrm{firstdom}(p) \cap X$.

(64) For every composable sequence p holds $\mathrm{dom}\,\mathrm{compose}_{\mathrm{firstdom}(p)}\, p = \mathrm{firstdom}(p)$.

(65) For every composable sequence p and for every function f such that $\mathrm{rng}\, f \subseteq \mathrm{firstdom}(p)$ holds $\langle f \rangle \frown p$ is a composable sequence.

(66) For every composable sequence p and for every function f such that $\mathrm{lastrng}(p) \subseteq \mathrm{dom}\, f$ holds $p \frown \langle f \rangle$ is a composable sequence.

(67) For every composable sequence p such that $x \in \mathrm{firstdom}(p)$ and $x \in X$ holds $(\mathrm{apply}(p,x))(\mathrm{len}\, p+1) = (\mathrm{compose}_X\, p)(x)$.

Let X, Y be sets. Let us assume that if Y is empty, then X is empty. A composable sequence is said to be a composable sequence from X into Y if:

(Def. 9) $\mathrm{firstdom}(\mathrm{it}) = X$ and $\mathrm{lastrng}(\mathrm{it}) \subseteq Y$.

Let Y be a non empty set, let X be a set, and let F be a composable sequence from X into Y. Then $\mathrm{compose}_X\, F$ is a function from X into Y.

Let q be a non-empty non empty finite sequence. A finite sequence is called a composable sequence along q if:

(Def. 10) $\mathrm{len}\,\mathrm{it} + 1 = \mathrm{len}\, q$ and for every natural number i such that $i \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{it}(i) \in q(i+1)^{q(i)}$.

Let q be a non-empty non empty finite sequence. Note that every composable sequence along q is composable and non-empty.

One can prove the following two propositions:

(68) Let q be a non-empty non empty finite sequence and p be a composable sequence along q. If $p \neq \emptyset$, then $\mathrm{firstdom}(p) = q(1)$ and $\mathrm{lastrng}(p) \subseteq q(\mathrm{len}\, q)$.

(69) Let q be a non-empty non empty finite sequence and p be a composable sequence along q. Then $\mathrm{dom}\,\mathrm{compose}_{q(1)}\, p = q(1)$ and $\mathrm{rng}\,\mathrm{compose}_{q(1)}\, p \subseteq q(\mathrm{len}\, q)$.

Let f be a function and let n be an element of $\mathbb{N}$. The functor f^n yielding a function is defined by the condition (Def. 11).

(Def. 11) There exists a function p from $\mathbb{N}$ into $(\mathrm{dom}\, f \cup \mathrm{rng}\, f) \dot{\rightarrow} (\mathrm{dom}\, f \cup \mathrm{rng}\, f)$ such that $f^n = p(n)$ and $p(0) = \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}$ and for every element k of $\mathbb{N}$ there exists a function g such that $g = p(k)$ and $p(k+1) = g \cdot f$.

In the sequel m, n are natural numbers.
We now state a number of propositions:

(70) $f^0 = \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}$.

(71) $f^{n+1} = f^n \cdot f$.

(72) $f^1 = f$.

(73) $f^{n+1} = f \cdot f^n$.

(74) $\mathrm{dom}(f^n) \subseteq \mathrm{dom}\, f \cup \mathrm{rng}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{dom}\, f \cup \mathrm{rng}\, f$.

(75) If $n \neq 0$, then $\mathrm{dom}(f^n) \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{rng}\, f$.

(76) If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$, then $\mathrm{dom}(f^n) = \mathrm{dom}\, f$ and $\mathrm{rng}(f^n) \subseteq \mathrm{dom}\, f$.

(77) $f^n \cdot \mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f} = f^n$.

(78) $\mathrm{id}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f} \cdot f^n = f^n$.

(79) $f^n \cdot f^m = f^{n+m}$.

(80) If $n \neq 0$, then $(f^m)^n = f^{m \cdot n}$.

(81) If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$, then $(f^m)^n = f^{m \cdot n}$.

(82) $\emptyset^n = \emptyset$.

(83) $(\mathrm{id}_X)^n = \mathrm{id}_X$.

(84) If $\mathrm{rng}\, f$ misses $\mathrm{dom}\, f$, then $f^2 = \emptyset$.

(85) For every function f from X into X holds f^n is a function from X into X.

(86) For every function f from X into X holds $f^0 = \mathrm{id}_X$.

(87) For every function f from X into X holds $(f^m)^n = f^{m \cdot n}$.

(88) For every partial function f from X to X holds f^n is a partial function from X to X.

(89) If $n \neq 0$ and $a \in X$ and $f = X \longmapsto a$, then $f^n = f$.

(90) For every function f and for every natural number n holds $f^n = \mathrm{compose}_{\mathrm{dom}\, f \cup \mathrm{rng}\, f}(n \mapsto f)$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/card_1.html.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[3] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_5.html.

[4] Grzegorz Bancerek. Reduction relations. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/rewrite1.html.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[6] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[7] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[8] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[9] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[10] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[11] Czesław Byliński. A classical first order language. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/cqc_lang.html.

[12] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[13] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_4.html.

[14] Czesław Byliński. Cartesian categories. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/cat_4.html`.

[15] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[16] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pralg_1.html`.

[17] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[18] Dariusz Surowik. Cyclic groups and some of their properties — part I. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/gr_cy_1.html`.

[19] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funcop_1.html`.

[20] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[22] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[23] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[24] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[25] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[26] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[27] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received January 12, 1996

Published January 2, 2004