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Real Functions Spaces[1]

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Summary. This abstract contains a construction of the domain of functions defined in an arbitrary nonempty set, with values being real numbers. In every such set of functions we introduce several algebraic operations, which yield in this set the structures of a real linear space, of a ring, and of a real algebra. Formal definitions of such concepts are given.

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The articles [9], [4], [13], [11], [14], [2], [3], [1], [8], [7], [12], [5], [6], and [10] provide the notation and terminology for this paper.

We adopt the following convention: x_1, x_2, z are sets and A, B are non empty sets.

Let us consider A, B, let F be a binary operation on B^A, and let f, g be elements of B^A. Then $F(f,g)$ is an element of B^A.

Let A, B, C, D be non empty sets, let F be a function from $[:C,D:]$ into B^A, and let c_1 be an element of $[:C,D:]$. Then $F(c_1)$ is an element of B^A.

Let A, B be non empty sets and let f be a function from A into B. The functor ${}^@f$ yielding an element of B^A is defined as follows:

(Def. 1) ${}^@f = f$.

In the sequel f, g, h are elements of $\mathbb{R}^A$.

Let X, Z be non empty sets, let F be a binary operation on X, and let f, g be functions from Z into X. Then $F^\circ(f,g)$ is an element of X^Z.

Let X, Z be non empty sets, let F be a binary operation on X, let a be an element of X, and let f be a function from Z into X. Then $F^\circ(a,f)$ is an element of X^Z.

Let us consider A. The functor $+_{\mathbb{R}^A}$ yielding a binary operation on $\mathbb{R}^A$ is defined as follows:

(Def. 2) For all elements f, g of $\mathbb{R}^A$ holds $+_{\mathbb{R}^A}(f,g) = (+_{\mathbb{R}})^\circ(f,g)$.

Let us consider A. The functor $\cdot_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:

(Def. 3) For all elements f, g of $\mathbb{R}^A$ holds $\cdot_{\mathbb{R}^A}(f,g) = (\cdot_{\mathbb{R}})^\circ(f,g)$.

Let us consider A. The functor $\cdot^{\mathbb{R}}_{\mathbb{R}^A}$ yielding a function from $[:\mathbb{R},\mathbb{R}^A:]$ into $\mathbb{R}^A$ is defined by:

(Def. 4) For every real number a and for every element f of $\mathbb{R}^A$ and for every element x of A holds $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle)(x) = a \cdot f(x)$.

Let us consider A. The functor $\mathbf{0}_{\mathbb{R}^A}$ yielding an element of $\mathbb{R}^A$ is defined as follows:

(Def. 5) $\mathbf{0}_{\mathbb{R}^A} = A \longmapsto 0$.

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Let us consider A. The functor $\mathbf{1}_{\mathbb{R}^A}$ yields an element of $\mathbb{R}^A$ and is defined by:

(Def. 6) $\mathbf{1}_{\mathbb{R}^A} = A \longmapsto 1$.

The following propositions are true:

(10)[1] $h = +_{\mathbb{R}^A}(f, g)$ iff for every element x of A holds $h(x) = f(x) + g(x)$.

(11) $h = \cdot_{\mathbb{R}^A}(f, g)$ iff for every element x of A holds $h(x) = f(x) \cdot g(x)$.

(12) For every element x of A holds $\mathbf{1}_{\mathbb{R}^A}(x) = 1$.

(13) For every element x of A holds $\mathbf{0}_{\mathbb{R}^A}(x) = 0$.

(14) $\mathbf{0}_{\mathbb{R}^A} \neq \mathbf{1}_{\mathbb{R}^A}$.

In the sequel a, b are real numbers.
We now state the proposition

(15) $h = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle)$ iff for every element x of A holds $h(x) = a \cdot f(x)$.

In the sequel u, v, w denote vectors of $\langle \mathbb{R}^A, \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A} \rangle$.
Next we state a number of propositions:

(16) $+_{\mathbb{R}^A}(f, g) = +_{\mathbb{R}^A}(g, f)$.

(17) $+_{\mathbb{R}^A}(f, +_{\mathbb{R}^A}(g, h)) = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(f, g), h)$.

(18) $\cdot_{\mathbb{R}^A}(f, g) = \cdot_{\mathbb{R}^A}(g, f)$.

(19) $\cdot_{\mathbb{R}^A}(f, \cdot_{\mathbb{R}^A}(g, h)) = \cdot_{\mathbb{R}^A}(\cdot_{\mathbb{R}^A}(f, g), h)$.

(20) $\cdot_{\mathbb{R}^A}(\mathbf{1}_{\mathbb{R}^A}, f) = f$.

(21) $+_{\mathbb{R}^A}(\mathbf{0}_{\mathbb{R}^A}, f) = f$.

(22) $+_{\mathbb{R}^A}(f, \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle -1, f \rangle)) = \mathbf{0}_{\mathbb{R}^A}$.

(23) $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle 1, f \rangle) = f$.

(24) $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, f \rangle) \rangle) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a \cdot b, f \rangle)$.

(25) $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, f \rangle)) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a + b, f \rangle)$.

(26) $\cdot_{\mathbb{R}^A}(f, +_{\mathbb{R}^A}(g, h)) = +_{\mathbb{R}^A}(\cdot_{\mathbb{R}^A}(f, g), \cdot_{\mathbb{R}^A}(f, h))$.

(27) $\cdot_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), g) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, \cdot_{\mathbb{R}^A}(f, g) \rangle)$.

(28) There exist f, g such that

(i) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$, and

(ii) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ and if $z \neq x_1$, then $g(z) = 1$.

(29) Suppose that

(i) $x_1 \in A$,

(ii) $x_2 \in A$,

(iii) $x_1 \neq x_2$,

(iv) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$, and

(v) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ and if $z \neq x_1$, then $g(z) = 1$.

Let given a, b. If $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)) = \mathbf{0}_{\mathbb{R}^A}$, then $a = 0$ and $b = 0$.

[1] The propositions (1)–(9) have been removed.

(30) If $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then there exist f, g such that for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$.

(31) Suppose that

(i) $A = \{x_1, x_2\}$,

(ii) $x_1 \neq x_2$,

(iii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$, and

(iv) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ and if $z \neq x_1$, then $g(z) = 1$.

Let given h. Then there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle))$.

(32) If $A = \{x_1, x_2\}$ and $x_1 \neq x_2$, then there exist f, g such that for every h there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle))$.

(33) Suppose $A = \{x_1, x_2\}$ and $x_1 \neq x_2$. Then there exist f, g such that for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and for every h there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle))$.

(34) $\langle \mathbb{R}^A, \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}\rangle$ is a real linear space.

Let us consider A. The functor $\mathbb{R}^A_{\mathbb{R}}$ yields a strict real linear space and is defined as follows:

(Def. 7) $\mathbb{R}^A_{\mathbb{R}} = \langle \mathbb{R}^A, \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}\rangle$.

We now state the proposition

(37)[2] There exists a strict real linear space V and there exist vectors u, v of V such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$ and for every vector w of V there exist a, b such that $w = a \cdot u + b \cdot v$.

Let us consider A. The functor $\mathrm{RRing} A$ yields a strict double loop structure and is defined as follows:

(Def. 12)[3] $\mathrm{RRing} A = \langle \mathbb{R}^A, +_{\mathbb{R}^A}, \cdot_{\mathbb{R}^A}, \mathbf{1}_{\mathbb{R}^A}, \mathbf{0}_{\mathbb{R}^A}\rangle$.

Let us consider A. Observe that $\mathrm{RRing} A$ is non empty.

We now state the proposition

(42)[4] Let x, y, z be elements of $\mathrm{RRing} A$. Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RRing} A} = x$ and there exists an element t of $\mathrm{RRing} A$ such that $x + t = 0_{\mathrm{RRing} A}$ and $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RRing} A} = x$ and $\mathbf{1}_{\mathrm{RRing} A} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

Let us mention that there exists a non empty double loop structure which is strict, Abelian, add-associative, right zeroed, right complementable, associative, commutative, right unital, and right distributive.

A ring is an Abelian add-associative right zeroed right complementable associative left unital right unital distributive non empty double loop structure.

We now state the proposition

(43) $\mathrm{RRing} A$ is a commutative ring.

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[2] The propositions (35) and (36) have been removed.
[3] The definitions (Def. 8)–(Def. 11) have been removed.
[4] The propositions (38)–(41) have been removed.

We introduce algebra structures which are extensions of double loop structure and RLS structure and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $[:\mathbb{R}, \text{the carrier}:]$ into the carrier, and the unity and the zero are elements of the carrier.

Let us note that there exists an algebra structure which is non empty.

Let us consider A. The functor RAlgebra$A$ yielding a strict algebra structure is defined as follows:

(Def. 19)[5] $\text{RAlgebra}A = \langle \mathbb{R}^A, \cdot_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}, \mathbf{1}_{\mathbb{R}^A}, \mathbf{0}_{\mathbb{R}^A} \rangle$.

Let us consider A. One can verify that RAlgebra$A$ is non empty.

One can prove the following proposition

(49)[6] Let x, y, z be elements of RAlgebra$A$ and given a, b. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{\text{RAlgebra}A} = x$ and there exists an element t of RAlgebra$A$ such that $x+t = 0_{\text{RAlgebra}A}$ and $x\cdot y = y\cdot x$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ and $x\cdot \mathbf{1}_{\text{RAlgebra}A} = x$ and $x\cdot(y+z) = x\cdot y + x\cdot z$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $a\cdot(x+y) = a\cdot x + a\cdot y$ and $(a+b)\cdot x = a\cdot x + b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$.

Let I_1 be a non empty algebra structure. We say that I_1 is algebra-like if and only if the condition (Def. 20) is satisfied.

(Def. 20) Let x, y, z be elements of I_1 and given a, b. Then $x\cdot \mathbf{1}_{(I_1)} = x$ and $x\cdot(y+z) = x\cdot y + x\cdot z$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $a\cdot(x+y) = a\cdot x + a\cdot y$ and $(a+b)\cdot x = a\cdot x + b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$.

One can verify that there exists a non empty algebra structure which is strict, Abelian, add-associative, right zeroed, right complementable, commutative, associative, and algebra-like.

An algebra is an Abelian add-associative right zeroed right complementable commutative associative algebra-like non empty algebra structure.

The following proposition is true

(50) RAlgebra$A$ is an algebra.

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