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Binary Operations Applied to Functions

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Summary. In the article we introduce functors yielding to a binary operation its composition with an arbitrary functions on its left side, its right side or both. We prove theorems describing the basic properties of these functors. We introduce also constant functions and converse of a function. The recent concept is defined for an arbitrary function, however is meaningful in the case of functions which range is a subset of a Cartesian product of two sets. Then the converse of a function has the same domain as the function itself and assigns to an element of the domain the mirror image of the ordered pair assigned by the function. In the case of functions defined on a non-empty set we redefine the above mentioned functors and prove simplified versions of theorems proved in the general case. We prove also theorems stating relationships between introduced concepts and such properties of binary operations as commutativity or associativity.

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The articles [7], [6], [9], [8], [11], [10], [3], [5], [4], [1], and [2] provide the notation and terminology for this paper.

The following proposition is true

(1) For every binary relation R and for all sets A, B such that $A \neq \emptyset$ and $B \neq \emptyset$ and $R = [:A, B:]$ holds $\mathrm{dom}\, R = A$ and $\mathrm{rng}\, R = B$.

In the sequel f, g, h are functions and A is a set.

Next we state three propositions:

(2) $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$.

(3) If $\mathrm{dom}\, f = \mathrm{dom}\, g$, then $\mathrm{dom}(f \cdot h) = \mathrm{dom}(g \cdot h)$.

(4) If $\mathrm{dom}\, f = \emptyset$ and $\mathrm{dom}\, g = \emptyset$, then $f = g$.

We adopt the following convention: F denotes a function and B, x, y, z denote sets.

Let us consider f. The functor $f^{\smile}$ yielding a function is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{dom}(f^{\smile}) = \mathrm{dom}\, f$, and

(ii) for every x such that $x \in \mathrm{dom}\, f$ holds for all y, z such that $f(x) = \langle y, z \rangle$ holds $f^{\smile}(x) = \langle z, y \rangle$ but $f(x) = f^{\smile}(x)$ or there exist y, z such that $f(x) = \langle y, z \rangle$.

Next we state several propositions:

(6)[1] $\langle f, g \rangle = \langle g, f \rangle^{\smile}$.

[1] The proposition (5) has been removed.

(7) $(f \restriction A)^\smile = f^\smile \restriction A$.

(8) $(f^\smile)^\smile = f$.

(9) $(\delta_A)^\smile = \delta_A$.

(10) $\langle f, g \rangle \restriction A = \langle f \restriction A, g \rangle$.

(11) $\langle f, g \rangle \restriction A = \langle f, g \restriction A \rangle$.

Let A, z be sets. The functor $A \longmapsto z$ yields a set and is defined by:

(Def. 2) $A \longmapsto z = [: A, \{z\} :]$.

Let A, z be sets. One can check that $A \longmapsto z$ is function-like and relation-like.
One can prove the following propositions:

(13)[2] If $x \in A$, then $(A \longmapsto z)(x) = z$.

(14) If $A \neq \emptyset$, then $\mathrm{rng}(A \longmapsto x) = \{x\}$.

(15) If $\mathrm{rng}\, f = \{x\}$, then $f = \mathrm{dom}\, f \longmapsto x$.

(16) $\mathrm{dom}(\emptyset \longmapsto x) = \emptyset$ and $\mathrm{rng}(\emptyset \longmapsto x) = \emptyset$.

(17) If for every z such that $z \in \mathrm{dom}\, f$ holds $f(z) = x$, then $f = \mathrm{dom}\, f \longmapsto x$.

(18) $(A \longmapsto x) \restriction B = A \cap B \longmapsto x$.

(19) $\mathrm{dom}(A \longmapsto x) = A$ and $\mathrm{rng}(A \longmapsto x) \subseteq \{x\}$.

(20) If $x \in B$, then $(A \longmapsto x)^{-1}(B) = A$.

(21) $(A \longmapsto x)^{-1}(\{x\}) = A$.

(22) If $x \notin B$, then $(A \longmapsto x)^{-1}(B) = \emptyset$.

(23) If $x \in \mathrm{dom}\, h$, then $h \cdot (A \longmapsto x) = A \longmapsto h(x)$.

(24) If $A \neq \emptyset$ and $x \in \mathrm{dom}\, h$, then $\mathrm{dom}(h \cdot (A \longmapsto x)) \neq \emptyset$.

(25) $(A \longmapsto x) \cdot h = h^{-1}(A) \longmapsto x$.

(26) $(A \longmapsto \langle x, y \rangle)^\smile = A \longmapsto \langle y, x \rangle$.

Let us consider F, f, g. The functor $F^\circ(f, g)$ yielding a set is defined as follows:

(Def. 3) $F^\circ(f, g) = F \cdot \langle f, g \rangle$.

Let us consider F, f, g. Observe that $F^\circ(f, g)$ is function-like and relation-like.
One can prove the following propositions:

(27) For every h such that $\mathrm{dom}\, h = \mathrm{dom}(F^\circ(f, g))$ and for every set z such that $z \in \mathrm{dom}(F^\circ(f, g))$ holds $h(z) = F(f(z), g(z))$ holds $h = F^\circ(f, g)$.

(28) If $x \in \mathrm{dom}(F^\circ(f, g))$, then $(F^\circ(f, g))(x) = F(f(x), g(x))$.

(29) If $f \restriction A = g \restriction A$, then $F^\circ(f, h) \restriction A = F^\circ(g, h) \restriction A$.

(30) If $f \restriction A = g \restriction A$, then $F^\circ(h, f) \restriction A = F^\circ(h, g) \restriction A$.

(31) $F^\circ(f, g) \cdot h = F^\circ(f \cdot h, g \cdot h)$.

(32) $h \cdot F^\circ(f, g) = (h \cdot F)^\circ(f, g)$.

[2] The proposition (12) has been removed.

Let us consider F, f, x. The functor $F^{\circ}(f,x)$ yields a set and is defined by:

(Def. 4) $F^{\circ}(f,x) = F \cdot \langle f, \mathrm{dom}\, f \longmapsto x \rangle$.

Let us consider F, f, x. Note that $F^{\circ}(f,x)$ is function-like and relation-like.

We now state several propositions:

(34)[3] $F^{\circ}(f,x) = F^{\circ}(f, \mathrm{dom}\, f \longmapsto x)$.

(35) If $x \in \mathrm{dom}(F^{\circ}(f,z))$, then $(F^{\circ}(f,z))(x) = F(f(x), z)$.

(36) If $f \restriction A = g \restriction A$, then $F^{\circ}(f,x) \restriction A = F^{\circ}(g,x) \restriction A$.

(37) $F^{\circ}(f,x) \cdot h = F^{\circ}(f \cdot h, x)$.

(38) $h \cdot F^{\circ}(f,x) = (h \cdot F)^{\circ}(f,x)$.

(39) $F^{\circ}(f,x) \cdot \mathrm{id}_A = F^{\circ}(f \restriction A, x)$.

Let us consider F, x, g. The functor $F^{\circ}(x,g)$ yielding a set is defined by:

(Def. 5) $F^{\circ}(x,g) = F \cdot \langle \mathrm{dom}\, g \longmapsto x, g \rangle$.

Let us consider F, x, g. Observe that $F^{\circ}(x,g)$ is function-like and relation-like.

One can prove the following propositions:

(41)[4] $F^{\circ}(x,g) = F^{\circ}(\mathrm{dom}\, g \longmapsto x, g)$.

(42) If $x \in \mathrm{dom}(F^{\circ}(z, f))$, then $(F^{\circ}(z, f))(x) = F(z, f(x))$.

(43) If $f \restriction A = g \restriction A$, then $F^{\circ}(x, f) \restriction A = F^{\circ}(x,g) \restriction A$.

(44) $F^{\circ}(x, f) \cdot h = F^{\circ}(x, f \cdot h)$.

(45) $h \cdot F^{\circ}(x, f) = (h \cdot F)^{\circ}(x, f)$.

(46) $F^{\circ}(x, f) \cdot \mathrm{id}_A = F^{\circ}(x, f \restriction A)$.

For simplicity, we adopt the following rules: X, Y are non empty sets, F is a binary operation on X, f, g, h are functions from Y into X, and x, x_1, x_2 are elements of X.

One can prove the following proposition

(47) $F^{\circ}(f, g)$ is a function from Y into X.

Let X, Z be non empty sets, let F be a binary operation on X, and let f, g be functions from Z into X. Then $F^{\circ}(f, g)$ is a function from Z into X.

We now state a number of propositions:

(48) For every element z of Y holds $(F^{\circ}(f, g))(z) = F(f(z), g(z))$.

(49) For every function h from Y into X such that for every element z of Y holds $h(z) = F(f(z), g(z))$ holds $h = F^{\circ}(f, g)$.

(51)[5] For every function g from X into X holds $F^{\circ}(\mathrm{id}_X, g) \cdot f = F^{\circ}(f, g \cdot f)$.

(52) For every function g from X into X holds $F^{\circ}(g, \mathrm{id}_X) \cdot f = F^{\circ}(g \cdot f, f)$.

(53) $F^{\circ}(\mathrm{id}_X, \mathrm{id}_X) \cdot f = F^{\circ}(f, f)$.

(54) For every function g from X into X holds $(F^{\circ}(\mathrm{id}_X, g))(x) = F(x, g(x))$.

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[3] The proposition (33) has been removed.
[4] The proposition (40) has been removed.
[5] The proposition (50) has been removed.

(55) For every function g from X into X holds $(F^\circ(g, \mathrm{id}_X))(x) = F(g(x), x)$.

(56) $(F^\circ(\mathrm{id}_X, \mathrm{id}_X))(x) = F(x, x)$.

(57) For all A, B and for every set x such that $x \in B$ holds $A \longmapsto x$ is a function from A into B.

(58) For all A, X, x holds $A \longmapsto x$ is a function from A into X.

(59) $F^\circ(f, x)$ is a function from Y into X.

Let X, Z be non empty sets, let F be a binary operation on X, let f be a function from Z into X, and let x be an element of X. Then $F^\circ(f, x)$ is a function from Z into X.

One can prove the following propositions:

(60) For every element y of Y holds $(F^\circ(f, x))(y) = F(f(y), x)$.

(61) If for every element y of Y holds $g(y) = F(f(y), x)$, then $g = F^\circ(f, x)$.

(63)[6] $F^\circ(\mathrm{id}_X, x) \cdot f = F^\circ(f, x)$.

(64) $(F^\circ(\mathrm{id}_X, x))(x) = F(x, x)$.

(65) $F^\circ(x, g)$ is a function from Y into X.

Let X, Z be non empty sets, let F be a binary operation on X, let x be an element of X, and let g be a function from Z into X. Then $F^\circ(x, g)$ is a function from Z into X.

Next we state several propositions:

(66) For every element y of Y holds $(F^\circ(x, f))(y) = F(x, f(y))$.

(67) If for every element y of Y holds $g(y) = F(x, f(y))$, then $g = F^\circ(x, f)$.

(69)[7] $F^\circ(x, \mathrm{id}_X) \cdot f = F^\circ(x, f)$.

(70) $(F^\circ(x, \mathrm{id}_X))(x) = F(x, x)$.

(71) Let X, Y, Z be non empty sets, f be a function from X into $[:Y, Z:]$, and x be an element of X. Then $f^\smile(x) = \langle f(x)_{\mathbf{2}}, f(x)_{\mathbf{1}} \rangle$.

(72) For all non empty sets X, Y, Z and for every function f from X into $[:Y, Z:]$ holds $\mathrm{rng}\, f$ is a relation between Y and Z.

Let X, Y, Z be non empty sets and let f be a function from X into $[:Y, Z:]$. Then $\mathrm{rng}\, f$ is a relation between Y and Z.

Let X, Y, Z be non empty sets and let f be a function from X into $[:Y, Z:]$. Then $f^\smile$ is a function from X into $[:Z, Y:]$.

We now state the proposition

(73) For all non empty sets X, Y, Z and for every function f from X into $[:Y, Z:]$ holds $\mathrm{rng}(f^\smile) = (\mathrm{rng}\, f)^\smile$.

In the sequel y denotes an element of Y.

The following propositions are true:

(74) If F is associative, then $F^\circ(F^\circ(x_1, f), x_2) = F^\circ(x_1, F^\circ(f, x_2))$.

(75) If F is associative, then $F^\circ(F^\circ(f, x), g) = F^\circ(f, F^\circ(x, g))$.

(76) If F is associative, then $F^\circ(F^\circ(f, g), h) = F^\circ(f, F^\circ(g, h))$.

(77) If F is associative, then $F^\circ(F(x_1, x_2), f) = F^\circ(x_1, F^\circ(x_2, f))$.

[6] The proposition (62) has been removed.
[7] The proposition (68) has been removed.

(78) If F is associative, then $F^\circ(f, F(x_1, x_2)) = F^\circ(F^\circ(f, x_1), x_2)$.

(79) If F is commutative, then $F^\circ(x, f) = F^\circ(f, x)$.

(80) If F is commutative, then $F^\circ(f, g) = F^\circ(g, f)$.

(81) If F is idempotent, then $F^\circ(f, f) = f$.

(82) If F is idempotent, then $(F^\circ(f(y), f))(y) = f(y)$.

(83) If F is idempotent, then $(F^\circ(f, f(y)))(y) = f(y)$.

(84) For all functions F, f, g such that $[\!:\operatorname{rng} f, \operatorname{rng} g:\!] \subseteq \operatorname{dom} F$ holds $\operatorname{dom}(F^\circ(f, g)) = \operatorname{dom} f \cap \operatorname{dom} g$.

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