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Minimization of Finite State Machines[1]

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Summary. We have formalized deterministic finite state machines closely following the textbook [10], pp. 88–119 up to the minimization theorem. In places, we have changed the approach presented in the book as it turned out to be too specific and inconvenient. Our work also revealed several minor mistakes in the book. After defining a structure for an outputless finite state machine, we have derived the structures for the transition assigned output machine (Mealy) and state assigned output machine (Moore). The machines are then proved similar, in the sense that for any Mealy (Moore) machine there exists a Moore (Mealy) machine producing essentially the same response for the same input. The rest of work is then done for Mealy machines. Next, we define equivalence of machines, equivalence and *k*-equivalence of states, and characterize a process of constructing for a given Mealy machine, the machine equivalent to it in which no two states are equivalent. The final, minimization theorem states:

> **Theorem 4.5:** Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be reduced, connected finite-state machines. Then the state graphs of $\mathbf{M}_1$ and $\mathbf{M}_2$ are isomorphic if and only if $\mathbf{M}_1$ and $\mathbf{M}_2$ are equivalent.

and it is the last theorem in this article.

MML Identifier: `FSM_1`.

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The articles [15], [7], [19], [2], [17], [12], [9], [18], [16], [14], [20], [4], [6], [5], [8], [3], [1], [13], and [11] provide the notation and terminology for this paper.

1. PRELIMINARIES

In this paper m, n, i, k denote natural numbers.

One can prove the following propositions:

(1) For all natural numbers m, n such that $m < n$ there exists a natural number p such that $n = m + p$ and $1 \leq p$.

(2) If $i \in \mathrm{Seg}\,n$, then $i + m \in \mathrm{Seg}(n + m)$.

(3) If $i > 0$ and $i + m \in \mathrm{Seg}(n + m)$, then $i \in \mathrm{Seg}\,n$ and $i \in \mathrm{Seg}(n + m)$.

(4) If $k < i$, then there exists a natural number j such that $j = i - k$ and $1 \leq j$.

(5) Let D be a non empty set and d_1 be a finite sequence of elements of D. Suppose d_1 is non empty. Then there exists an element d of D and there exists a finite sequence d_2 of elements of D such that $d = d_1(1)$ and $d_1 = \langle d \rangle \frown d_2$.

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(6) For every finite sequence d_1 and for every set d such that $i \in \mathrm{dom}\, d_1$ holds $(\langle d \rangle \frown d_1)(i+1) = d_1(i)$.

(7) Let S be a set, D_1, D_2 be non empty sets, f_1 be a function from S into D_1, and f_2 be a function from D_1 into D_2. If f_1 is bijective and f_2 is bijective, then $f_2 \cdot f_1$ is bijective.

(8) For every set Y and for all equivalence relations E_1, E_2 of Y such that $\mathrm{Classes}\, E_1 = \mathrm{Classes}\, E_2$ holds $E_1 = E_2$.

(9) For every non empty set W holds every partition of W is non empty.

(10) For every finite set Z holds every partition of Z is finite.

Let W be a non empty set. Note that every partition of W is non empty.

Let Z be a finite set. Note that every partition of Z is finite.

Let X be a non empty finite set. Note that there exists a partition of X which is non empty and finite.

We follow the rules: X, A are non empty finite sets, P_1 is a partition of X, and P_2, P_3 are partitions of A.

We now state several propositions:

(11) Let X be a non empty set, P_1 be a partition of X, and P_4 be a set. If $P_4 \in P_1$, then there exists an element x of X such that $x \in P_4$.

(12) $\mathrm{card}\, P_1 \leq \mathrm{card}\, X$.

(13) If P_2 is finer than P_3, then $\mathrm{card}\, P_3 \leq \mathrm{card}\, P_2$.

(14) If P_2 is finer than P_3, then for every element p_2 of P_3 there exists an element p_1 of P_2 such that $p_1 \subseteq p_2$.

(15) If P_2 is finer than P_3 and $\mathrm{card}\, P_2 = \mathrm{card}\, P_3$, then $P_2 = P_3$.

2. Definitions and Terminology

Let I_1 be a set. We introduce FSM over I_1 which are extensions of 1-sorted structure and are systems

⟨ a carrier, a transition, an initial state ⟩,

where the carrier is a set, the transition is a function from $[\!:$ the carrier, $I_1 :\!]$ into the carrier, and the initial state is an element of the carrier.

Let I_1 be a set and let f_3 be a FSM over I_1. A state of f_3 is an element of f_3.

Let X be a set. Observe that there exists a FSM over X which is non empty and finite.

Let us note that there exists a 1-sorted structure which is finite and non empty.

Let S be a finite 1-sorted structure. One can check that the carrier of S is finite.

For simplicity, we adopt the following convention: I_1, O_1 are non empty sets, f_3 is a non empty FSM over I_1, s is an element of I_1, w, w_1, w_2 are finite sequences of elements of I_1, and q, q', q_1, q_2 are states of f_3.

Let I_1 be a non empty set, let f_3 be a non empty FSM over I_1, let s be an element of I_1, and let q be a state of f_3. The functor s-succ(q) yielding a state of f_3 is defined as follows:

(Def. 1) $s\text{-succ}(q) = (\text{the transition of } f_3)(\langle q, s \rangle)$.

Let I_1 be a non empty set, let f_3 be a non empty FSM over I_1, let q be a state of f_3, and let w be a finite sequence of elements of I_1. The functor (q,w)-admissible yields a finite sequence of elements of the carrier of f_3 and is defined by the conditions (Def. 2).

(Def. 2)(i) $(q,w)\text{-admissible}(1) = q$,

(ii) $\mathrm{len}((q,w)\text{-admissible}) = \mathrm{len}\, w + 1$, and

(iii) for every i such that $1 \leq i$ and $i \leq \mathrm{len}\, w$ there exists an element w_3 of I_1 and there exist states q_3, q_4 of f_3 such that $w_3 = w(i)$ and $q_3 = (q,w)\text{-admissible}(i)$ and $q_4 = (q,w)\text{-admissible}(i+1)$ and $w_3\text{-succ}(q_3) = q_4$.

We now state the proposition

(16) $(q, \varepsilon_{(I_1)})$-admissible $= \langle q \rangle$.

Let I_1 be a non empty set, let f_3 be a non empty FSM over I_1, let w be a finite sequence of elements of I_1, and let q_1, q_2 be states of f_3. The predicate $q_1 \xrightarrow{w} q_2$ is defined by:

(Def. 3) (q_1, w)-admissible$(\operatorname{len} w + 1) = q_2$.

The following proposition is true

(17) $q \xrightarrow{\varepsilon_{(I_1)}} q$.

Let I_1 be a non empty set, let f_3 be a non empty FSM over I_1, let w be a finite sequence of elements of I_1, and let q_5 be a finite sequence of elements of the carrier of f_3. We say that q_5 is admissible for w if and only if:

(Def. 4) There exists a state q_1 of f_3 such that $q_1 = q_5(1)$ and (q_1, w)-admissible $= q_5$.

Next we state the proposition

(18) $\langle q \rangle$ is admissible for $\varepsilon_{(I_1)}$.

Let us consider I_1, f_3, q, w. The functor w-succ(q) yields a state of f_3 and is defined by:

(Def. 5) $q \xrightarrow{w} w\text{-succ}(q)$.

Next we state several propositions:

(19) (q, w)-admissible$(\operatorname{len}((q, w)\text{-admissible})) = q'$ iff $q \xrightarrow{w} q'$.

(20) For every k such that $1 \leq k$ and $k \leq \operatorname{len} w_1$ holds $(q_1, w_1 \frown w_2)$-admissible$(k) = (q_1, w_1)$-admissible(k).

(21) If $q_1 \xrightarrow{w_1} q_2$, then $(q_1, w_1 \frown w_2)$-admissible$(\operatorname{len} w_1 + 1) = q_2$.

(22) If $q_1 \xrightarrow{w_1} q_2$, then for every k such that $1 \leq k$ and $k \leq \operatorname{len} w_2 + 1$ holds $(q_1, w_1 \frown w_2)$-admissible$(\operatorname{len} w_1 + k) = (q_2, w_2)$-admissible$(k)$.

(23) If $q_1 \xrightarrow{w_1} q_2$, then $(q_1, w_1 \frown w_2)$-admissible $= ((q_1, w_1)\text{-admissible}_{\restriction \operatorname{len} w_1 + 1}) \frown (q_2, w_2)$-admissible.

3. Mealy and Moore Machines

Let I_1 be a set and let O_1 be a non empty set. We consider Mealy-FSM over I_1, O_1 as extensions of FSM over I_1 as systems

⟨ a carrier, a transition, an output function, an initial state ⟩,

where the carrier is a set, the transition is a function from $[:$the carrier$, I_1:]$ into the carrier, the output function is a function from $[:$the carrier$, I_1:]$ into O_1, and the initial state is an element of the carrier.

We consider Moore-FSM over I_1, O_1 as extensions of FSM over I_1 as systems

⟨ a carrier, a transition, an output function, an initial state ⟩,

where the carrier is a set, the transition is a function from $[:$the carrier$, I_1:]$ into the carrier, the output function is a function from the carrier into O_1, and the initial state is an element of the carrier.

Let I_1 be a set, let X be a finite non empty set, let T be a function from $[:X, I_1:]$ into X, and let I be an element of X. Observe that $\langle X, T, I \rangle$ is finite and non empty.

Let I_1 be a set, let O_1 be a non empty set, let X be a finite non empty set, let T be a function from $[:X, I_1:]$ into X, let O be a function from $[:X, I_1:]$ into O_1, and let I be an element of X. Note that Mealy-FSM$\langle X, T, O, I \rangle$ is finite and non empty.

Let I_1 be a set, let O_1 be a non empty set, let X be a finite non empty set, let T be a function from $[:X, I_1:]$ into X, let O be a function from X into O_1, and let I be an element of X. Note that $\langle X, T, O, I \rangle$ is finite and non empty.

Let I_1 be a set and let O_1 be a non empty set. Note that there exists a Mealy-FSM over I_1, O_1 which is finite and non empty and there exists a Moore-FSM over I_1, O_1 which is finite and non empty.

For simplicity, we adopt the following rules: t_1, t_2, t_3, t_4 denote non empty Mealy-FSM over I_1, O_1, s_1 denotes a non empty Moore-FSM over I_1, O_1, q_6 denotes a state of s_1, q, q_1, q_2, q_7, q_8, q_9, q_{10}, q'_1, q_{11}, q_{12}, q_{13} denote states of t_1, q_{14}, q_{15} denote states of t_2, and q_{21}, q_{22} denote states of t_3.

Let us consider I_1, O_1, t_1, q_{11}, w. The functor (q_{11}, w)-response yields a finite sequence of elements of O_1 and is defined as follows:

(Def. 6) $\mathrm{len}((q_{11}, w)\text{-response}) = \mathrm{len}\, w$ and for every i such that $i \in \mathrm{dom}\, w$ holds $(q_{11}, w)\text{-response}(i) =$ (the output function of t_1)$(\langle\langle (q_{11}, w)\text{-admissible}(i), w(i)\rangle\rangle)$.

We now state the proposition

(24) $(q_{11}, \varepsilon_{(I_1)})\text{-response} = \varepsilon_{(O_1)}$.

Let us consider I_1, O_1, s_1, q_6, w. The functor (q_6, w)-response yields a finite sequence of elements of O_1 and is defined by:

(Def. 7) $\mathrm{len}((q_6, w)\text{-response}) = \mathrm{len}\, w + 1$ and for every i such that $i \in \mathrm{Seg}(\mathrm{len}\, w + 1)$ holds $(q_6, w)\text{-response}(i) =$ (the output function of s_1)$((q_6, w)\text{-admissible}(i))$.

One can prove the following three propositions:

(25) $(q_6, w)\text{-response}(1) =$ (the output function of s_1)(q_6).

(26) If $q_{12} \xrightarrow{w_1} q_{13}$, then $(q_{12}, w_1 \frown w_2)\text{-response} = (q_{12}, w_1)\text{-response} \frown (q_{13}, w_2)\text{-response}$.

(27) If $q_{14} \xrightarrow{w_1} q_{15}$ and $q_{21} \xrightarrow{w_1} q_{22}$ and $(q_{15}, w_2)\text{-response} \neq (q_{22}, w_2)\text{-response}$, then $(q_{14}, w_1 \frown w_2)\text{-response} \neq (q_{21}, w_1 \frown w_2)\text{-response}$.

In the sequel O_2 denotes a finite non empty set, t_5 denotes a finite non empty Mealy-FSM over I_1, O_2, and s_2 denotes a finite non empty Moore-FSM over I_1, O_2.

Let us consider I_1, O_1, let t_1 be a non empty Mealy-FSM over I_1, O_1, and let s_1 be a non empty Moore-FSM over I_1, O_1. We say that t_1 is similar to s_1 if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let t_6 be a finite sequence of elements of I_1. Then $\langle$(the output function of s_1)(the initial state of s_1)$\rangle \frown$ (the initial state of t_1, t_6)-response = (the initial state of s_1, t_6)-response.

The following two propositions are true:

(28) For every non empty finite Moore-FSM s_1 over I_1, O_1 holds there exists a non empty finite Mealy-FSM over I_1, O_1 which is similar to s_1.

(29) There exists s_2 such that t_5 is similar to s_2.

4. Equivalence of States and Machines

Let I_1, O_1 be non empty sets and let t_2, t_3 be non empty Mealy-FSM over I_1, O_1. We say that t_2 and t_3 are equivalent if and only if:

(Def. 9) For every finite sequence w of elements of I_1 holds (the initial state of t_2, w)-response = (the initial state of t_3, w)-response.

Let us notice that the predicate t_2 and t_3 are equivalent is reflexive and symmetric.

Next we state the proposition

(30) If t_2 and t_3 are equivalent and t_3 and t_4 are equivalent, then t_2 and t_4 are equivalent.

Let us consider I_1, O_1, t_1, q_8, q_9. We say that q_8 and q_9 are equivalent if and only if:

(Def. 10) For every w holds (q_8, w)-response $=$ (q_9, w)-response.

Let us notice that the predicate q_8 and q_9 are equivalent is reflexive and symmetric.

Next we state several propositions:

(33)[1] If q_1 and q_2 are equivalent and q_2 and q_7 are equivalent, then q_1 and q_7 are equivalent.

(34) If $q'_1 =$ (the transition of t_1)($\langle q_8, s\rangle$), then for every i such that $i \in \mathrm{Seg}(\mathrm{len}\, w + 1)$ holds $(q_8, \langle s\rangle \frown w)$-admissible$(i+1) = (q'_1, w)$-admissible$(i)$.

(35) If $q'_1 =$ (the transition of t_1)($\langle q_8, s\rangle$), then $(q_8, \langle s\rangle \frown w)$-response $= \langle$(the output function of t_1)($\langle q_8, s\rangle$)$\rangle \frown (q'_1, w)$-response.

(36) q_8 and q_9 are equivalent if and only if for every s holds (the output function of t_1)($\langle q_8, s\rangle$) $=$ (the output function of t_1)($\langle q_9, s\rangle$) and (the transition of t_1)($\langle q_8, s\rangle$) and (the transition of t_1)($\langle q_9, s\rangle$) are equivalent.

(37) Suppose q_8 and q_9 are equivalent. Let given w, i. Suppose $i \in \mathrm{dom}\, w$. Then there exist states q_{16}, q_{17} of t_1 such that $q_{16} = (q_8, w)$-admissible(i) and $q_{17} = (q_9, w)$-admissible(i) and q_{16} and q_{17} are equivalent.

Let us consider I_1, O_1, t_1, q_8, q_9, k. We say that q_8 and q_9 are k-equivalent if and only if:

(Def. 11) For every w such that $\mathrm{len}\, w \leq k$ holds (q_8, w)-response $=$ (q_9, w)-response.

The following propositions are true:

(38) q_8 and q_8 are k-equivalent.

(39) If q_8 and q_9 are k-equivalent, then q_9 and q_8 are k-equivalent.

(40) If q_8 and q_9 are k-equivalent and q_9 and q_{10} are k-equivalent, then q_8 and q_{10} are k-equivalent.

(41) If q_8 and q_9 are equivalent, then q_8 and q_9 are k-equivalent.

(42) q_8 and q_9 are 0-equivalent.

(43) If q_8 and q_9 are $k+m$-equivalent, then q_8 and q_9 are k-equivalent.

(44) Suppose $1 \leq k$. Then q_8 and q_9 are k-equivalent if and only if the following conditions are satisfied:

(i) q_8 and q_9 are 1-equivalent, and

(ii) for every element s of I_1 and for every natural number k_1 such that $k_1 = k - 1$ holds (the transition of t_1)($\langle q_8, s\rangle$) and (the transition of t_1)($\langle q_9, s\rangle$) are k_1-equivalent.

Let us consider I_1, O_1, t_1, i. The functor i-EqS-Rel(t_1) yields an equivalence relation of the carrier of t_1 and is defined by:

(Def. 12) For all q_8, q_9 holds $\langle q_8, q_9\rangle \in i$-EqS-Rel$(t_1)$ iff q_8 and q_9 are i-equivalent.

Let us consider I_1, O_1, let t_1 be a non empty Mealy-FSM over I_1, O_1, and let us consider i. The functor i-EqS-Part(t_1) yielding a non empty partition of the carrier of t_1 is defined as follows:

(Def. 13) i-EqS-Part$(t_1) =$ Classes(i-EqS-Rel(t_1)).

The following propositions are true:

(45) $(k+1)$-EqS-Part(t_1) is finer than k-EqS-Part(t_1).

[1] The propositions (31) and (32) have been removed.

(46) If Classes(k-EqS-Rel(t_1)) $=$ Classes(($k+1$)-EqS-Rel(t_1)), then for every m holds Classes(($k+m$)-EqS-Rel(t_1)) $=$ Classes(k-EqS-Rel(t_1)).

(47) If k-EqS-Part(t_1) $=$ ($k+1$)-EqS-Part(t_1), then for every m holds ($k+m$)-EqS-Part(t_1) $=$ k-EqS-Part(t_1).

(48) If ($k+1$)-EqS-Part(t_1) $\neq$ k-EqS-Part(t_1), then for every i such that $i \leq k$ holds ($i+1$)-EqS-Part(t_1) $\neq$ i-EqS-Part(t_1).

(49) For every finite non empty Mealy-FSM t_1 over I_1, O_1 holds k-EqS-Part(t_1) $=$ ($k+1$)-EqS-Part(t_1) or card(k-EqS-Part(t_1)) $<$ card(($k+1$)-EqS-Part(t_1)).

(50) $[q]_{0\text{-EqS-Rel}(t_1)} =$ the carrier of t_1.

(51) 0-EqS-Part(t_1) $=$ {the carrier of t_1}.

(52) For every finite non empty Mealy-FSM t_1 over I_1, O_1 such that $n+1=$ card(the carrier of t_1) holds ($n+1$)-EqS-Part(t_1) $=$ n-EqS-Part(t_1).

Let us consider I_1, O_1, let t_1 be a non empty Mealy-FSM over I_1, O_1, and let I_2 be a partition of the carrier of t_1. We say that I_2 is final if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let q_8, q_9 be states of t_1. Then q_8 and q_9 are equivalent if and only if there exists an element X of I_2 such that $q_8 \in X$ and $q_9 \in X$.

Next we state three propositions:

(53) If k-EqS-Part(t_1) is final, then ($k+1$)-EqS-Rel(t_1) $=$ k-EqS-Rel(t_1).

(54) k-EqS-Part(t_1) $=$ ($k+1$)-EqS-Part(t_1) iff k-EqS-Part(t_1) is final.

(55) Let t_1 be a finite non empty Mealy-FSM over I_1, O_1. Suppose $n+1=$ card(the carrier of t_1). Then there exists a natural number k such that $k \leq n$ and k-EqS-Part(t_1) is final.

Let us consider I_1, O_1 and let t_1 be a finite non empty Mealy-FSM over I_1, O_1. The functor final-Partition(t_1) yields a partition of the carrier of t_1 and is defined as follows:

(Def. 15) final-Partition(t_1) is final.

Next we state the proposition

(56) For every finite non empty Mealy-FSM t_1 over I_1, O_1 such that $n+1=$ card(the carrier of t_1) holds final-Partition(t_1) $=$ n-EqS-Part(t_1).

5. The Reduction of a Mealy Machine

In the sequel t_1, r_1 denote finite non empty Mealy-FSM over I_1, O_1 and q denotes a state of t_1.

Let I_1, O_1 be non empty sets, let t_1 be a finite non empty Mealy-FSM over I_1, O_1, let q_{18} be an element of final-Partition(t_1), and let s be an element of I_1. The functor (s, q_{18})-C-succ yielding an element of final-Partition(t_1) is defined by the condition (Def. 16).

(Def. 16) There exists a state q of t_1 and there exists a natural number n such that $q \in q_{18}$ and $n+1=$ card(the carrier of t_1) and (s, q_{18})-C-succ $=$ [(the transition of t_1)($\langle q, s\rangle$)]$_{n\text{-EqS-Rel}(t_1)}$.

Let us consider I_1, O_1, t_1, let q_{18} be an element of final-Partition(t_1), and let us consider s. The functor (q_{18}, s)-C-response yields an element of O_1 and is defined by:

(Def. 17) There exists q such that $q \in q_{18}$ and (q_{18}, s)-C-response $=$ (the output function of t_1)($\langle q, s\rangle$).

Let us consider I_1, O_1, t_1. The reduction of t_1 yielding a strict Mealy-FSM over I_1, O_1 is defined by the conditions (Def. 18).

(Def. 18)(i) The carrier of the reduction of $t_1 =$ final-Partition(t_1),

(ii) for every state Q of the reduction of t_1 and for every s and for every state q of t_1 such that $q \in Q$ holds (the transition of $t_1)(\langle q, s\rangle) \in$ (the transition of the reduction of $t_1)(\langle Q, s\rangle)$ and (the output function of $t_1)(\langle q, s\rangle) =$ (the output function of the reduction of $t_1)(\langle Q, s\rangle)$, and

(iii) the initial state of $t_1 \in$ the initial state of the reduction of t_1.

Let us consider I_1, O_1, t_1. One can check that the reduction of t_1 is non empty and finite.

We now state two propositions:

(57) Let q_{19} be a state of r_1. Suppose $r_1 =$ the reduction of t_1 and $q \in q_{19}$. Let k be a natural number. If $k \in \mathrm{Seg}(\mathrm{len}\, w+1)$, then (q,w)-admissible$(k) \in (q_{19},w)$-admissible(k).

(58) t_1 and the reduction of t_1 are equivalent.

6. MACHINE ISOMORPHISM

In the sequel q_{20}, q_{23} are states of r_1 and T_1 is a function from the carrier of t_2 into the carrier of t_3.

Let us consider I_1, O_1, t_2, t_3. We say that t_2 and t_3 are isomorphic if and only if the condition (Def. 19) is satisfied.

(Def. 19) There exists T_1 such that

(i) T_1 is bijective,

(ii) T_1(the initial state of $t_2) =$ the initial state of t_3, and

(iii) for all q_{14}, s holds T_1((the transition of $t_2)(\langle q_{14}, s\rangle)) =$ (the transition of $t_3)(\langle T_1(q_{14}), s\rangle)$ and (the output function of $t_2)(\langle q_{14}, s\rangle) =$ (the output function of $t_3)(\langle T_1(q_{14}), s\rangle)$.

Let us notice that the predicate t_2 and t_3 are isomorphic is reflexive and symmetric.

We now state four propositions:

(59) If t_2 and t_3 are isomorphic and t_3 and t_4 are isomorphic, then t_2 and t_4 are isomorphic.

(60) Suppose that for every state q of t_2 and for every s holds T_1((the transition of $t_2)(\langle q, s\rangle)) =$ (the transition of $t_3)(\langle T_1(q), s\rangle)$. Let given k. If $1 \leq k$ and $k \leq \mathrm{len}\, w+1$, then $T_1((q_{14},w)$-admissible$(k)) = (T_1(q_{14}),w)$-admissible(k).

(61) Suppose that

(i) T_1(the initial state of $t_2) =$ the initial state of t_3, and

(ii) for every state q of t_2 and for every s holds T_1((the transition of $t_2)(\langle q, s\rangle)) =$ (the transition of $t_3)(\langle T_1(q), s\rangle)$ and (the output function of $t_2)(\langle q, s\rangle) =$ (the output function of $t_3)(\langle T_1(q), s\rangle)$.

Then q_{14} and q_{15} are equivalent if and only if $T_1(q_{14})$ and $T_1(q_{15})$ are equivalent.

(62) If $r_1 =$ the reduction of t_1 and $q_{20} \neq q_{23}$, then q_{20} and q_{23} are not equivalent.

7. REDUCED AND CONNECTED MACHINES

Let I_1, O_1 be non empty sets and let I_2 be a non empty Mealy-FSM over I_1, O_1. We say that I_2 is reduced if and only if:

(Def. 20) For all states q_8, q_9 of I_2 such that $q_8 \neq q_9$ holds q_8 and q_9 are not equivalent.

One can prove the following proposition

(63) The reduction of t_1 is reduced.

Let us consider I_1, O_1. Observe that there exists a non empty Mealy-FSM over I_1, O_1 which is reduced and finite.

In the sequel R_1 denotes a reduced finite non empty Mealy-FSM over I_1, O_1.

We now state two propositions:

(64) R_1 and the reduction of R_1 are isomorphic.

(65) t_1 is reduced if and only if there exists a finite non empty Mealy-FSM M over I_1, O_1 such that t_1 and the reduction of M are isomorphic.

Let us consider I_1, O_1, let t_1 be a non empty Mealy-FSM over I_1, O_1, and let I_2 be a state of t_1. We say that I_2 is accessible if and only if:

(Def. 21) There exists w such that the initial state of $t_1 \xrightarrow{w} I_2$.

Let us consider I_1, O_1 and let I_2 be a non empty Mealy-FSM over I_1, O_1. We say that I_2 is connected if and only if:

(Def. 22) Every state of I_2 is accessible.

Let us consider I_1, O_1. Observe that there exists a finite non empty Mealy-FSM over I_1, O_1 which is connected.

In the sequel C_1, C_2, C_3 are connected finite non empty Mealy-FSM over I_1, O_1.

Next we state the proposition

(66) The reduction of C_1 is connected.

Let us consider I_1, O_1. Note that there exists a non empty Mealy-FSM over I_1, O_1 which is connected, reduced, and finite.

Let us consider I_1, O_1 and let t_1 be a non empty Mealy-FSM over I_1, O_1. The functor accessible-States(t_1) is defined by:

(Def. 23) accessible-States$(t_1) = \{q; q$ ranges over states of t_1: q is accessible$\}$.

Let us consider I_1, O_1, t_1. One can check that accessible-States(t_1) is finite and non empty.

One can prove the following propositions:

(67) accessible-States$(t_1) \subseteq$ the carrier of t_1 and for every q holds $q \in$ accessible-States(t_1) iff q is accessible.

(68) (The transition of $t_1) \restriction [\!:$accessible-States$(t_1), I_1 :\!]$ is a function from $[\!:$accessible-States(t_1), $I_1 :\!]$ into accessible-States(t_1).

(69) Let c_1 be a function from $[\!:$accessible-States$(t_1), I_1 :\!]$ into accessible-States(t_1), c_2 be a function from $[\!:$accessible-States$(t_1), I_1 :\!]$ into O_1, and c_3 be an element of accessible-States(t_1). Suppose $c_1 =$ (the transition of $t_1) \restriction [\!:$accessible-States$(t_1), I_1 :\!]$ and $c_2 =$ (the output function of $t_1) \restriction [\!:$accessible-States$(t_1), I_1 :\!]$ and $c_3 =$ the initial state of t_1. Then t_1 and Mealy-FSM$\langle$accessible-States$(t_1), c_1, c_2, c_3\rangle$ are equivalent.

(70) There exists C_1 such that

(i) the transition of $C_1 =$ (the transition of $t_1) \restriction [\!:$accessible-States$(t_1), I_1 :\!]$,

(ii) the output function of $C_1 =$ (the output function of $t_1) \restriction [\!:$accessible-States$(t_1), I_1 :\!]$,

(iii) the initial state of $C_1 =$ the initial state of t_1, and

(iv) t_1 and C_1 are equivalent.

8. Machine Union

Let I_1 be a set, let O_1 be a non empty set, and let t_2, t_3 be non empty Mealy-FSM over I_1, O_1. The functor Mealy-U(t_2,t_3) yields a strict Mealy-FSM over I_1, O_1 and is defined by the conditions (Def. 24).

(Def. 24)(i) The carrier of Mealy-U(t_2,t_3) = (the carrier of t_2) $\cup$ (the carrier of t_3),

(ii) the transition of Mealy-U(t_2,t_3) = (the transition of t_2)$+\cdot$(the transition of t_3),

(iii) the output function of Mealy-U(t_2,t_3) = (the output function of t_2)$+\cdot$(the output function of t_3), and

(iv) the initial state of Mealy-U(t_2,t_3) = the initial state of t_2.

Let I_1 be a set, let O_1 be a non empty set, and let t_2, t_3 be non empty finite Mealy-FSM over I_1, O_1. One can check that Mealy-U(t_2,t_3) is non empty and finite.

We now state four propositions:

(71) If t_1 = Mealy-U(t_2,t_3) and the carrier of t_2 misses the carrier of t_3 and $q_{14} = q$, then (q_{14},w)-admissible $= (q,w)$-admissible.

(72) If t_1 = Mealy-U(t_2,t_3) and the carrier of t_2 misses the carrier of t_3 and $q_{14} = q$, then (q_{14},w)-response $= (q,w)$-response.

(73) If t_1 = Mealy-U(t_2,t_3) and the carrier of t_2 misses the carrier of t_3 and $q_{21} = q$, then (q_{21},w)-admissible $= (q,w)$-admissible.

(74) If t_1 = Mealy-U(t_2,t_3) and the carrier of t_2 misses the carrier of t_3 and $q_{21} = q$, then (q_{21},w)-response $= (q,w)$-response.

In the sequel R_2, R_3 denote reduced non empty Mealy-FSM over I_1, O_1.

We now state the proposition

(75) Suppose t_1 = Mealy-U(R_2,R_3) and the carrier of R_2 misses the carrier of R_3 and R_2 and R_3 are equivalent. Then there exists a state Q of the reduction of t_1 such that

(i) the initial state of $R_2 \in Q$,

(ii) the initial state of $R_3 \in Q$, and

(iii) Q = the initial state of the reduction of t_1.

In the sequel C_4, C_5 are connected reduced non empty Mealy-FSM over I_1, O_1 and q_{24}, q_{25} are states of t_1.

The following two propositions are true:

(76) Let c_{11}, c_{12} be states of C_4. Suppose that

(i) $c_{11} = q_{24}$,

(ii) $c_{12} = q_{25}$,

(iii) the carrier of C_4 misses the carrier of C_5,

(iv) C_4 and C_5 are equivalent,

(v) t_1 = Mealy-U(C_4,C_5), and

(vi) c_{11} and c_{12} are not equivalent.

Then q_{24} and q_{25} are not equivalent.

(77) Let c_{21}, c_{22} be states of C_5. Suppose that

(i) $c_{21} = q_{24}$,

(ii) $c_{22} = q_{25}$,

(iii) the carrier of C_4 misses the carrier of C_5,

(iv) C_4 and C_5 are equivalent,

(v) $t_1 = \text{Mealy-U}(C_4, C_5)$, and

(vi) c_{21} and c_{22} are not equivalent.

Then q_{24} and q_{25} are not equivalent.

In the sequel C_4, C_5 denote connected reduced finite non empty Mealy-FSM over I_1, O_1.
We now state three propositions:

(78) Suppose the carrier of C_4 misses the carrier of C_5 and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there do not exist elements q_1, q_2 of Q such that $q_1 \in$ the carrier of C_4 and $q_2 \in$ the carrier of C_4 and $q_1 \neq q_2$.

(79) Suppose the carrier of C_4 misses the carrier of C_5 and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there do not exist elements q_1, q_2 of Q such that $q_1 \in$ the carrier of C_5 and $q_2 \in$ the carrier of C_5 and $q_1 \neq q_2$.

(80) Suppose the carrier of C_4 misses the carrier of C_5 and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there exist elements q_1, q_2 of Q such that $q_1 \in$ the carrier of C_4 and $q_2 \in$ the carrier of C_5 and for every element q of Q holds $q = q_1$ or $q = q_2$.

9. THE MINIMIZATION THEOREM

We now state several propositions:

(81) Let t_2, t_3 be finite non empty Mealy-FSM over I_1, O_1. Then there exist finite non empty Mealy-FSM f_4, f_5 over I_1, O_1 such that the carrier of f_4 misses the carrier of f_5 and f_4 and t_2 are isomorphic and f_5 and t_3 are isomorphic.

(82) If t_2 and t_3 are isomorphic, then t_2 and t_3 are equivalent.

(83) Suppose the carrier of C_4 misses the carrier of C_5 and C_4 and C_5 are equivalent. Then C_4 and C_5 are isomorphic.

(84) If C_2 and C_3 are equivalent, then the reduction of C_2 and the reduction of C_3 are isomorphic.

(85) Let M_1, M_2 be connected reduced finite non empty Mealy-FSM over I_1, O_1. Then M_1 and M_2 are isomorphic if and only if M_1 and M_2 are equivalent.

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