JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

Full Subtracter Circuit. Part II

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Summary. In this article we continue investigations from [22] of verification of a design of subtracter circuit. We define it as a combination of multi cell circuit using schemes from [6]. As the main result we prove the stability of the circuit.

MML Identifier: FSCIRC_2.

WWW: http://mizar.org/JFM/Vol15/fscirc_2.html

The articles [16], [15], [21], [20], [2], [17], [24], [1], [8], [9], [4], [10], [3], [18], [25], [14], [19], [12], [13], [11], [23], [5], [7], and [22] provide the notation and terminology for this paper.

Let n be a natural number and let x, y be finite sequences. The functor n-BitSubtracterStr(x,y) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by the condition (Def. 1).

(Def. 1) There exist many sorted sets f, g indexed by $\mathbb{N}$ such that

- (i) n-BitSubtracterStr$(x,y) = f(n)$,
- (ii) $f(0) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,
- (iii) $g(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$, and
- (iv) for every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = g(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = $ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number and let x, y be finite sequences. The functor n-BitSubtracterCirc(x,y) yielding a Boolean strict circuit of n-BitSubtracterStr(x,y) with denotation held in gates is defined by the condition (Def. 2).

(Def. 2) There exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that

- (i) n-BitSubtracterStr$(x,y) = f(n)$,
- (ii) n-BitSubtracterCirc$(x,y) = g(n)$,
- (iii) $f(0) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,
- (iv) $g(0) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,
- (v) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$, and
- (vi) for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z =$

$h(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) =$ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number and let x, y be finite sequences. The functor n-BitBorrowOutput(x, y) yields an element of InnerVertices(n-BitSubtracterStr(x, y)) and is defined by the condition (Def. 3).

(Def. 3) There exists a many sorted set h indexed by $\mathbb{N}$ such that n-BitBorrowOutput$(x, y) = h(n)$ and $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$ and for every natural number n holds $h(n+1) =$ BorrowOutput$(x(n+1), y(n+1), h(n))$.

We now state several propositions:

(1) Let x, y be finite sequences and f, g, h be many sorted sets indexed by $\mathbb{N}$. Suppose that

(i) $f(0) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(ii) $g(0) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(iii) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$, and

(iv) for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) =$ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number. Then n-BitSubtracterStr$(x, y) = f(n)$ and n-BitSubtracterCirc$(x, y) = g(n)$ and n-BitBorrowOutput$(x, y) = h(n)$.

(2) For all finite sequences a, b holds 0-BitSubtracterStr$(a, b) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$ and 0-BitSubtracterCirc$(a, b) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$ and 0-BitBorrowOutput$(a, b) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$.

(3) Let a, b be finite sequences and c be a set. Suppose $c = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$. Then 1-BitSubtracterStr$(a, b) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}) +\cdot$ BitSubtracterWithBorrowStr$(a(1), b(1), c)$ and 1-BitSubtracterCirc$(a, b) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}) +\cdot$ BitSubtracterWithBorrowCirc$(a(1), b(1), c)$ and 1-BitBorrowOutput$(a, b) =$ BorrowOutput$(a(1), b(1), c)$.

(4) For all sets a, b, c such that $c = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$ holds 1-BitSubtracterStr$(\langle a \rangle, \langle b \rangle) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}) +\cdot$ BitSubtracterWithBorrowStr(a, b, c) and 1-BitSubtracterCirc$(\langle a \rangle, \langle b \rangle) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}) +\cdot$ BitSubtracterWithBorrowCirc(a, b, c) and 1-BitBorrowOutput$(\langle a \rangle, \langle b \rangle) =$ BorrowOutput(a, b, c).

(5) Let n be a natural number, p, q be finite sequences with length n, and p_1, p_2, q_1, q_2 be finite sequences. Then n-BitSubtracterStr$(p \frown p_1, q \frown q_1) = n$-BitSubtracterStr$(p \frown p_2, q \frown q_2)$ and n-BitSubtracterCirc$(p \frown p_1, q \frown q_1) = n$-BitSubtracterCirc$(p \frown p_2, q \frown q_2)$ and n-BitBorrowOutput$(p \frown p_1, q \frown q_1) = n$-BitBorrowOutput$(p \frown p_2, q \frown q_2)$.

(6) Let n be a natural number, x, y be finite sequences with length n, and a, b be sets. Then $(n+1)$-BitSubtracterStr$(x \frown \langle a \rangle, y \frown \langle b \rangle) = (n$-BitSubtracterStr$(x, y)) +\cdot$ BitSubtracterWithBorrowStr$(a, b, n$-BitBorrowOutp and $(n+1)$-BitSubtracterCirc$(x \frown \langle a \rangle, y \frown \langle b \rangle) = (n$-BitSubtracterCirc$(x, y)) +\cdot$ BitSubtracterWithBorrowCirc$(a, b, n$-Bit and $(n+1)$-BitBorrowOutput$(x \frown \langle a \rangle, y \frown \langle b \rangle) =$ BorrowOutput$(a, b, n$-BitBorrowOutput$(x, y))$.

(7) Let n be a natural number and x, y be finite sequences. Then $(n+1)$-BitSubtracterStr$(x, y) = (n$-BitSubtracterStr$(x, y)) +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitSubtracterCirc$(x, y) = (n$-BitSubtracterCirc$(x, y)) +\cdot$ BitSubtracter $1), y(n+1), n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitBorrowOutput$(x, y) =$ BorrowOutput$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$.

(8) For all natural numbers n, m such that $n \leq m$ and for all finite sequences x, y holds InnerVertices(n-BitSubtracterStr(x, y)) $\subseteq$ InnerVertices(m-BitSubtracterStr(x, y)).

(9) For every natural number n and for all finite sequences x, y holds InnerVertices($(n+1)$-BitSubtracterStr$(x,y)) =$ InnerVertices(n-BitSubtracterStr$(x,y)) \cup$ InnerVertices(BitSubtracterWithBorrowStr$(x(n+1),y(n+1),n$-BitBorrowOutput$(x,y)))$.

Let k, n be natural numbers. Let us assume that $k \geq 1$ and $k \leq n$. Let x, y be finite sequences. The functor (k,n)-BitSubtracterOutput(x,y) yields an element of InnerVertices(n-BitSubtracterStr(x,y)) and is defined as follows:

(Def. 4) There exists a natural number i such that $k = i+1$ and (k,n)-BitSubtracterOutput$(x,y) =$ BitSubtracterOutput$(x(k),y(k),i$-BitBorrowOutput$(x,y))$.

One can prove the following propositions:

(10) For all natural numbers n, k such that $k < n$ and for all finite sequences x, y holds $(k+1,n)$-BitSubtracterOutput$(x,y) =$ BitSubtracterOutput$(x(k+1),y(k+1),k$-BitBorrowOutput$(x,y))$.

(11) For every natural number n and for all finite sequences x, y holds InnerVertices(n-BitSubtracterStr(x,y)) is a binary relation.

(12) For all sets x, y, c holds InnerVertices(BorrowIStr$(x,y,c)) = \{\langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y,c\rangle, \mathrm{and}_2\rangle, \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle\}$.

(13) For all sets x, y, c such that $x \neq \langle\langle y,c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BorrowIStr$(x,y,c)) = \{x,y,c\}$.

(14) For all sets x, y, c holds InnerVertices(BorrowStr$(x,y,c)) = \{\langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y,c\rangle, \mathrm{and}_2\rangle, \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle\} \cup \{$BorrowOutput$(x,y,c)\}$.

(15) For all sets x, y, c such that $x \neq \langle\langle y,c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BorrowStr$(x,y,c)) = \{x,y,c\}$.

(16) For all sets x, y, c such that $x \neq \langle\langle y,c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{xor}\rangle$ holds InputVertices(BitSubtracterWithBorrowStr$(x,y,c)) = \{x,y,c\}$.

(17) For all sets x, y, c holds InnerVertices(BitSubtracterWithBorrowStr$(x,y,c)) = \{\langle\langle x,y\rangle, \mathrm{xor}\rangle$, 2GatesCircOutput$(x,y,c,\mathrm{xor})\} \cup \{\langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y,c\rangle, \mathrm{and}_2\rangle, \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle\} \cup \{$BorrowOutput$(x,y,c)\}$.

Let n be a natural number and let x, y be finite sequences. Observe that n-BitBorrowOutput(x,y) is pair.

Next we state several propositions:

(18) Let x, y be finite sequences and n be a natural number. Then $(n$-BitBorrowOutput$(x,y))_{\mathbf{1}} = \varepsilon$ and $(n$-BitBorrowOutput$(x,y))_{\mathbf{2}} = \mathit{Boolean}^0 \longmapsto \mathit{true}$ and $\pi_1((n$-BitBorrowOutput$(x,y))_{\mathbf{2}}) = \mathit{Boolean}^0$ or $\overline{\overline{(n\text{-BitBorrowOutput}(x,y))_{\mathbf{1}}}} = 3$ and $(n$-BitBorrowOutput$(x,y))_{\mathbf{2}} = \mathrm{or}_3$ and $\pi_1((n$-BitBorrowOutput$(x,y))_{\mathbf{2}}) = \mathit{Boolean}^3$.

(19) Let n be a natural number, x, y be finite sequences, and p be a set. Then n-BitBorrowOutput$(x,y) \neq \langle p, \mathrm{and}_2\rangle$ and n-BitBorrowOutput$(x,y) \neq \langle p, \mathrm{and}_{2a}\rangle$ and n-BitBorrowOutput$(x,y) \neq \langle p, \mathrm{xor}\rangle$.

(20) Let f, g be nonpair yielding finite sequences and n be a natural number. Then InputVertices($(n+1)$-BitSubtracterStr$(f,g)) =$ InputVertices(n-BitSubtracterStr$(f,g)) \cup$ (InputVertices(BitSubtracterWithBorrowStr$(f(n+1),g(n+1),n$-BitBorrowOutput$(f,g))) \setminus \{n$-BitBorrowOutput$(f,g)\}$) and InnerVertices(n-BitSubtracterStr(f,g)) is a binary relation and InputVertices(n-BitSubtracterStr(f,g)) has no pairs.

(21) For every natural number n and for all nonpair yielding finite sequences x, y with length n holds InputVertices(n-BitSubtracterStr$(x,y)) = \mathrm{rng}\, x \cup \mathrm{rng}\, y$.

(22) Let x, y, c be sets, s be a state of $\mathrm{BorrowCirc}(x,y,c)$, and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle)$ and $a_2 = s(\langle\langle y,c\rangle, \mathrm{and}_2\rangle)$ and $a_3 = s(\langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle)$, then $(\mathrm{Following}(s))(\mathrm{BorrowOutput}(x,y,c)) = a_1 \vee a_2 \vee a_3$.

(23) Let x, y, c be sets. Suppose $x \neq \langle\langle y,c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{xor}\rangle$. Let s be a state of $\mathrm{BorrowCirc}(x,y,c)$. Then $\mathrm{Following}(s,2)$ is stable.

(24) Let x, y, c be sets. Suppose $x \neq \langle\langle y,c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{xor}\rangle$. Let s be a state of $\mathrm{BitSubtracterWithBorrowCirc}(x,y,c)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$. Then $(\mathrm{Following}(s,2))(\mathrm{BitSubtracterOutput}(x,y,c)) = a_1 \oplus a_2 \oplus a_3$ and $(\mathrm{Following}(s,2))(\mathrm{BorrowOutput}(x,y,c)) = \neg a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee \neg a_1 \wedge a_3$.

(25) Let x, y, c be sets. Suppose $x \neq \langle\langle y,c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x,c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x,y\rangle, \mathrm{xor}\rangle$. Let s be a state of $\mathrm{BitSubtracterWithBorrowCirc}(x,y,c)$. Then $\mathrm{Following}(s,2)$ is stable.

(26) Let n be a natural number, x, y be nonpair yielding finite sequences with length n, and s be a state of n-$\mathrm{BitSubtracterCirc}(x,y)$. Then $\mathrm{Following}(s, 1+2\cdot n)$ is stable.

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Received February 25, 2003

Published January 2, 2004