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# First-countable, Sequential, and Frechet Spaces

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**Summary.** This article contains a definition of three classes of topological spaces: first-countable, Frechet, and sequential. Next there are some facts about them, that every first-countable space is Frechet and every Frechet space is sequential. Next section contains a formalized construction of topological space which is Frechet but not first-countable. This article is based on [10, pp. 73–81].

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The articles [18], [21], [20], [11], [1], [13], [22], [5], [6], [17], [2], [3], [7], [16], [14], [8], [12], [4], [9], [15], and [19] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

One can prove the following proposition

(1) For every non empty 1-sorted structure $T$ and for every sequence $S$ of $T$ holds $\operatorname{rng} S$ is a subset of $T$.

Let $T$ be a non empty 1-sorted structure and let $S$ be a sequence of $T$. Then $\operatorname{rng} S$ is a subset of $T$.

The following two propositions are true:

(2) Let $T_1$ be a non empty 1-sorted structure, $T_2$ be a 1-sorted structure, and $S$ be a sequence of $T_1$. If $\operatorname{rng} S \subseteq$ the carrier of $T_2$, then $S$ is a sequence of $T_2$.

(3) For every non empty topological space $T$ and for every point $x$ of $T$ and for every basis $B$ of $x$ holds $B \neq \emptyset$.

Let $T$ be a non empty topological space and let $x$ be a point of $T$. Note that every basis of $x$ is non empty.

One can prove the following propositions:

(4) For every topological space $T$ and for all subsets $A$, $B$ of $T$ such that $A$ is open and $B$ is closed holds $A \setminus B$ is open.

(5) Let $T$ be a topological structure. Suppose that

(i) $\emptyset_T$ is closed,

(ii) $\Omega_T$ is closed,

(iii) for all subsets $A$, $B$ of $T$ such that $A$ is closed and $B$ is closed holds $A \cup B$ is closed, and

(iv) for every family $F$ of subsets of $T$ such that $F$ is closed holds $\bigcap F$ is closed.

Then $T$ is a topological space.

 

(6) Let $T$ be a topological space, $S$ be a non empty topological structure, and $f$ be a map from $T$ into $S$. Suppose that for every subset $A$ of $S$ holds $A$ is closed iff $f^{-1}(A)$ is closed. Then $S$ is a topological space.

(7) Let $x$ be a point of the metric space of real numbers and $x'$, $r$ be real numbers. If $x' = x$ and $r > 0$, then $\mathrm{Ball}(x,r) = ]x'-r, x'+r[$.

(8) Let $A$ be a subset of $\mathbb{R}^{\mathbf{1}}$. Then $A$ is open if and only if for every real number $x$ such that $x \in A$ there exists a real number $r$ such that $r > 0$ and $]x-r, x+r[ \subseteq A$.

(9) For every sequence $S$ of $\mathbb{R}^{\mathbf{1}}$ such that for every natural number $n$ holds $S(n) \in ]n-\frac{1}{4}, n+\frac{1}{4}[$ holds $\mathrm{rng}\, S$ is closed.

(10) For every subset $B$ of $\mathbb{R}^{\mathbf{1}}$ such that $B = \mathbb{N}$ holds $B$ is closed.

(11) Let $M$ be a non empty metric space, $x$ be a point of $M_{\mathrm{top}}$, and $x'$ be a point of $M$. Suppose $x = x'$. Then there exists a basis $B$ of $x$ such that

(i) $B = \{\mathrm{Ball}(x', \frac{1}{n}); n$ ranges over natural numbers: $n \neq 0\}$,

(ii) $B$ is countable, and

(iii) there exists a function $f$ from $\mathbb{N}$ into $B$ such that for every set $n$ such that $n \in \mathbb{N}$ there exists a natural number $n'$ such that $n = n'$ and $f(n) = \mathrm{Ball}(x', \frac{1}{n'+1})$.

(12) For all functions $f$, $g$ holds $\mathrm{rng}(f{+}\cdot g) = f^{\circ}(\mathrm{dom}\, f \setminus \mathrm{dom}\, g) \cup \mathrm{rng}\, g$.

(13) For all sets $A$, $B$ such that $B \subseteq A$ holds $(\mathrm{id}_A)^{\circ}B = B$.

(15)[1] For all sets $A$, $B$, $x$ holds $\mathrm{dom}(\mathrm{id}_A{+}\cdot(B \longmapsto x)) = A \cup B$.

(16) For all sets $A$, $B$, $x$ such that $B \neq \emptyset$ holds $\mathrm{rng}(\mathrm{id}_A{+}\cdot(B \longmapsto x)) = (A \setminus B) \cup \{x\}$.

(17) For all sets $A$, $B$, $C$, $x$ such that $C \subseteq A$ holds $(\mathrm{id}_A{+}\cdot(B \longmapsto x))^{-1}(C \setminus \{x\}) = C \setminus B \setminus \{x\}$.

(18) For all sets $A$, $B$, $x$ such that $x \notin A$ holds $(\mathrm{id}_A{+}\cdot(B \longmapsto x))^{-1}(\{x\}) = B$.

(19) For all sets $A$, $B$, $C$, $x$ such that $C \subseteq A$ and $x \notin A$ holds $(\mathrm{id}_A{+}\cdot(B \longmapsto x))^{-1}(C \cup \{x\}) = C \cup B$.

(20) For all sets $A$, $B$, $C$, $x$ such that $C \subseteq A$ and $x \notin A$ holds $(\mathrm{id}_A{+}\cdot(B \longmapsto x))^{-1}(C \setminus \{x\}) = C \setminus B$.

## 2. First-countable, Sequential, and Frechet Spaces

Let $T$ be a non empty topological structure. We say that $T$ is first-countable if and only if:

(Def. 1) For every point $x$ of $T$ holds there exists a basis of $x$ which is countable.

The following two propositions are true:

(21) For every non empty metric space $M$ holds $M_{\mathrm{top}}$ is first-countable.

(22) $\mathbb{R}^{\mathbf{1}}$ is first-countable.

Let us observe that $\mathbb{R}^{\mathbf{1}}$ is first-countable.

Let $T$ be a topological structure, let $S$ be a sequence of $T$, and let $x$ be a point of $T$. We say that $S$ is convergent to $x$ if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let $U_1$ be a subset of $T$. Suppose $U_1$ is open and $x \in U_1$. Then there exists a natural number $n$ such that for every natural number $m$ such that $n \leq m$ holds $S(m) \in U_1$.

The following proposition is true

(23) Let $T$ be a non empty topological structure, $x$ be a point of $T$, and $S$ be a sequence of $T$. If $S = \mathbb{N} \longmapsto x$, then $S$ is convergent to $x$.

[1] The proposition (14) has been removed.

Let $T$ be a topological structure and let $S$ be a sequence of $T$. We say that $S$ is convergent if and only if:

(Def. 3) There exists a point $x$ of $T$ such that $S$ is convergent to $x$.

Let $T$ be a non empty topological structure and let $S$ be a sequence of $T$. The functor $\mathrm{Lim}\,S$ yielding a subset of $T$ is defined as follows:

(Def. 4) For every point $x$ of $T$ holds $x \in \mathrm{Lim}\,S$ iff $S$ is convergent to $x$.

Let $T$ be a non empty topological structure. We say that $T$ is Frechet if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let $A$ be a subset of $T$ and $x$ be a point of $T$. If $x \in \overline{A}$, then there exists a sequence $S$ of $T$ such that $\mathrm{rng}\,S \subseteq A$ and $x \in \mathrm{Lim}\,S$.

Let $T$ be a non empty topological structure. We say that $T$ is sequential if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let $A$ be a subset of $T$. Then $A$ is closed if and only if for every sequence $S$ of $T$ such that $S$ is convergent and $\mathrm{rng}\,S \subseteq A$ holds $\mathrm{Lim}\,S \subseteq A$.

Next we state the proposition

(24) For every non empty topological space $T$ such that $T$ is first-countable holds $T$ is Frechet.

Let us observe that every non empty topological space which is first-countable is also Frechet.
The following propositions are true:

(26)[2] Let $T$ be a non empty topological space and $A$ be a subset of $T$. Suppose $A$ is closed. Let $S$ be a sequence of $T$. If $S$ is convergent and $\mathrm{rng}\,S \subseteq A$, then $\mathrm{Lim}\,S \subseteq A$.

(27) Let $T$ be a non empty topological space. Suppose that for every subset $A$ of $T$ such that for every sequence $S$ of $T$ such that $S$ is convergent and $\mathrm{rng}\,S \subseteq A$ holds $\mathrm{Lim}\,S \subseteq A$ holds $A$ is closed. Then $T$ is sequential.

(28) For every non empty topological space $T$ such that $T$ is Frechet holds $T$ is sequential.

Let us note that every non empty topological space which is Frechet is also sequential.

## 3. Counterexample of Frechet but Not First-countable Space

The strict non empty topological space $\mathbb{R}^1/_{\mathbb{N}}$ is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of $\mathbb{R}^1/_{\mathbb{N}} = (\mathbb{R} \setminus \mathbb{N}) \cup \{\mathbb{R}\}$, and

(ii) there exists a map $f$ from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^1/_{\mathbb{N}}$ such that $f = \mathrm{id}_{\mathbb{R}} +\cdot (\mathbb{N} \longmapsto \mathbb{R})$ and for every subset $A$ of $\mathbb{R}^1/_{\mathbb{N}}$ holds $A$ is closed iff $f^{-1}(A)$ is closed.

The following propositions are true:

(30)[3] $\mathbb{R}$ is a point of $\mathbb{R}^1/_{\mathbb{N}}$.

(31) Let $A$ be a subset of $\mathbb{R}^1/_{\mathbb{N}}$. Then $A$ is open and $\mathbb{R} \in A$ if and only if there exists a subset $O$ of $\mathbb{R}^{\mathbf{1}}$ such that $O$ is open and $\mathbb{N} \subseteq O$ and $A = (O \setminus \mathbb{N}) \cup \{\mathbb{R}\}$.

(32) For every set $A$ holds $A$ is a subset of $\mathbb{R}^1/_{\mathbb{N}}$ and $\mathbb{R} \notin A$ iff $A$ is a subset of $\mathbb{R}^{\mathbf{1}}$ and $\mathbb{N} \cap A = \emptyset$.

[2] The proposition (25) has been removed.
[3] The proposition (29) has been removed.

(33) Let $A$ be a subset of $\mathbb{R}^{\mathbf{1}}$ and $B$ be a subset of $\mathbb{R}^1/_{\mathbb{N}}$. If $A = B$, then $\mathbb{N} \cap A = \emptyset$ and $A$ is open iff $\mathbb{R} \notin B$ and $B$ is open.

(34) For every subset $A$ of $\mathbb{R}^1/_{\mathbb{N}}$ such that $A = \{\mathbb{R}\}$ holds $A$ is closed.

(35) $\mathbb{R}^1/_{\mathbb{N}}$ is not first-countable.

(36) $\mathbb{R}^1/_{\mathbb{N}}$ is Frechet.

(37) It is not true that for every non empty topological space $T$ such that $T$ is Frechet holds $T$ is first-countable.

## 4. Auxiliary Theorems

One can prove the following proposition

(40)[4] For every real number $r$ such that $r > 0$ there exists a natural number $n$ such that $\frac{1}{n} < r$ and $n \neq 0$.

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