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Function Domains and Frænkel Operator

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Summary. We deal with a non–empty set of functions and a non–empty set of functions from a set A to a non–empty set B. In the case when B is a non–empty set, B^A is redefined. It yields a non–empty set of functions from A to B. An element of such a set is redefined as a function from A to B. Some theorems concerning these concepts are proved, as well as a number of schemes dealing with infinity and the Axiom of Choice. The article contains a number of schemes allowing for simple logical transformations related to terms constructed with the Frænkel Operator.

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The articles [6], [3], [8], [9], [4], [7], [1], [2], and [5] provide the notation and terminology for this paper.

In this paper B is a non empty set and A, C, X are sets.

In this article we present several logical schemes. The scheme *Fraenkel5'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(v'); v'$ ranges over elements of $\mathcal{A} : \mathcal{P}[v']\} \subseteq \{\mathcal{F}(u'); u'$ ranges over elements of $\mathcal{A} : \mathcal{Q}[u']\}$

provided the following condition is satisfied:

- For every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{Q}[v]$.

The scheme *Fraenkel5”* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} \subseteq$
$\{\mathcal{F}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the parameters meet the following requirement:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u, v]$ holds $\mathcal{Q}[u, v]$.

The scheme *Fraenkel6'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{F}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[v_2]\}$

provided the following condition is satisfied:

- For every element v of $\mathcal{A}$ holds $\mathcal{P}[v]$ iff $\mathcal{Q}[v]$.

The scheme *Fraenkel6”* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} =$
$\{\mathcal{F}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the following condition is met:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ iff $\mathcal{Q}[u, v]$.

The scheme *FraenkelF'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_2]\}$

provided the parameters have the following property:

- For every element v of $\mathcal{A}$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF'R* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_2]\}$

provided the following condition is met:

- For every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{G}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_2, v_2]\}$

provided the following condition is satisfied:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{G}(u, v)$.

The scheme *FraenkelF6"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(v_2, u_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the parameters have the following properties:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ iff $\mathcal{Q}[u, v]$, and
- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{F}(v, u)$.

The following propositions are true:

(3)[1] Let A, B be non empty sets, F, G be functions from A into B, and X be a set. If $F \restriction X = G \restriction X$, then for every element x of A such that $x \in X$ holds $F(x) = G(x)$.

(5)[2] For all sets A, B holds $B^A \subseteq 2^{[:A, B:]}$.

(6) For all sets X, Y such that $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$ holds every element of Y^X is a partial function from A to B.

Now we present a number of schemes. The scheme *RelevantArgs* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a set $\mathcal{C}$, a function $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{B}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{D}(u'); u'$ ranges over elements of $\mathcal{A} : \mathcal{P}[u'] \wedge u' \in \mathcal{C}\} = \{\mathcal{E}(v'); v'$ ranges over elements of $\mathcal{A} : \mathcal{Q}[v'] \wedge v' \in \mathcal{C}\}$

provided the parameters meet the following requirements:

- $\mathcal{D} \restriction \mathcal{C} = \mathcal{E} \restriction \mathcal{C}$, and
- For every element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ holds $\mathcal{P}[u]$ iff $\mathcal{Q}[u]$.

The scheme *Fr Set0* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\{x; x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\} \subseteq \mathcal{A}$

for all values of the parameters.

The scheme *Gen1"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

For every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s, t]$ holds $\mathcal{Q}[\mathcal{F}(s, t)]$

provided the parameters satisfy the following condition:

- For every set s_1 such that $s_1 \in \{\mathcal{F}(s_2, t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_2, t_1]\}$ holds $\mathcal{Q}[s_1]$.

The scheme *Gen1"A* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

[1] The propositions (1) and (2) have been removed.
[2] The proposition (4) has been removed.

For every set s_1 such that $s_1 \in \{\mathcal{F}(s_2,t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_2,t_1]\}$ holds $\mathcal{Q}[s_1]$

provided the following condition is met:

- For every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ holds $\mathcal{Q}[\mathcal{F}(s,t)]$.

The scheme *Gen2"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a non empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{s_1; s_1$ ranges over elements of $\mathcal{C} : s_1 \in \{\mathcal{F}(s_2,t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_2,t_1]\} \wedge \mathcal{Q}[s_1]\} = \{\mathcal{F}(s_3,t_2); s_3$ ranges over elements of $\mathcal{A}, t_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_3,t_2] \wedge \mathcal{Q}[\mathcal{F}(s_3,t_2)]\}$

for all values of the parameters.

The scheme *Gen3'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s); s$ ranges over elements of $\mathcal{A} : s \in \{s_2; s_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[s_2]\} \wedge \mathcal{P}[s]\} = \{\mathcal{F}(s_3); s_3$ ranges over elements of $\mathcal{A} : \mathcal{Q}[s_3] \wedge \mathcal{P}[s_3]\}$

for all values of the parameters.

The scheme *Gen3"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : s \in \{s_2; s_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[s_2]\} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s_3,t_2); s_3$ ranges over elements of $\mathcal{A}, t_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[s_3] \wedge \mathcal{P}[s_3,t_2]\}$

for all values of the parameters.

The scheme *Gen4"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : \mathcal{P}[s,t]\} \subseteq \{\mathcal{F}(s_2,t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{Q}[s_2,t_1]\}$

provided the parameters meet the following condition:

- Let s be an element of $\mathcal{A}$ and t be an element of $\mathcal{B}$. If $\mathcal{P}[s,t]$, then there exists an element s' of $\mathcal{A}$ such that $\mathcal{Q}[s',t]$ and $\mathcal{F}(s,t) = \mathcal{F}(s',t)$.

The scheme *FrSet 1* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y); y$ ranges over elements of $\mathcal{A} : \mathcal{F}(y) \in \mathcal{B} \wedge \mathcal{P}[y]\} \subseteq \mathcal{B}$

for all values of the parameters.

The scheme *FrSet 2* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y); y$ ranges over elements of $\mathcal{A} : \mathcal{P}[y] \wedge \mathcal{F}(y) \notin \mathcal{B}\}$ misses $\mathcal{B}$

for all values of the parameters.

The scheme *FrEqua1* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, an element $\mathcal{C}$ of $\mathcal{B}$, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : \mathcal{Q}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}); s'$ ranges over elements of $\mathcal{A} : \mathcal{P}[s',\mathcal{C}]\}$

provided the following condition is satisfied:

- For every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ holds $\mathcal{Q}[s,t]$ iff $t = \mathcal{C}$ and $\mathcal{P}[s,t]$.

The scheme *FrEqua2* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, an element $\mathcal{C}$ of $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : t = \mathcal{C} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}); s'$ ranges over elements of $\mathcal{A} : \mathcal{P}[s',\mathcal{C}]\}$

for all values of the parameters.

Let I_1 be a set. We say that I_1 is functional if and only if:

(Def. 1) For every set x such that $x \in I_1$ holds x is a function.

One can verify that there exists a set which is non empty and functional.

Let P be a functional set. Observe that every element of P is function-like and relation-like.

The following proposition is true

(8)[3] For every function f holds $\{f\}$ is functional.

Let A, B be sets. Observe that B^A is functional.

Let A, B be sets. A functional non empty set is said to be a non empty set of functions from A to B if:

(Def. 2) Every element of it is a function from A into B.

We now state two propositions:

(10)[4] For every function f from A into C holds $\{f\}$ is a non empty set of functions from A to C.

(11) B^A is a non empty set of functions from A to B.

Let A be a set and let B be a non empty set. Then B^A is a non empty set of functions from A to B. Let F be a non empty set of functions from A to B. We see that the element of F is a function from A into B.

In the sequel p_1 is an element of B^A.

We now state two propositions:

(14)[5] Let X, Y be sets. Suppose $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$. Let f be an element of Y^X. Then there exists an element p_1 of B^A such that $p_1 \restriction X = f$.

(15) For every set X and for every p_1 holds $p_1 \restriction X = p_1 \restriction (A \cap X)$.

Now we present four schemes. The scheme *FraenkelFin* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(w); w$ ranges over elements of $\mathcal{A} : w \in \mathcal{B}\}$ is finite

provided the parameters meet the following requirement:

- $\mathcal{B}$ is finite.

The scheme *CartFin* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, sets $\mathcal{C}$, $\mathcal{D}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(u', v'); u'$ ranges over elements of $\mathcal{A}, v'$ ranges over elements of $\mathcal{B} : u' \in \mathcal{C} \wedge v' \in \mathcal{D}\}$ is finite

provided the parameters meet the following requirements:

- $\mathcal{C}$ is finite, and
- $\mathcal{D}$ is finite.

The scheme *Finiteness* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\operatorname{Fin}\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

Let x be an element of $\mathcal{A}$. Suppose $x \in \mathcal{B}$. Then there exists an element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ and $\mathcal{P}[y, x]$ and for every element z of $\mathcal{A}$ such that $z \in \mathcal{B}$ and $\mathcal{P}[z, y]$ holds $\mathcal{P}[y, z]$

provided the following conditions are satisfied:

- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$, and
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *Fin Im* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\operatorname{Fin}\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists an element c_1 of $\operatorname{Fin}\mathcal{A}$ such that for every element t of $\mathcal{A}$ holds $t \in c_1$ if and only if there exists an element t' of $\mathcal{B}$ such that $t' \in \mathcal{C}$ and $t = \mathcal{F}(t')$ and $\mathcal{P}[t, t']$

for all values of the parameters.

The following proposition is true

(16) For all sets A, B such that A is finite and B is finite holds B^A is finite.

[3] The proposition (7) has been removed.

[4] The proposition (9) has been removed.

[5] The propositions (12) and (13) have been removed.

Now we present three schemes. The scheme *ImFin* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a set $\mathcal{C}$, a set $\mathcal{D}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(p'_1); p'_1$ ranges over elements of $\mathcal{B}^{\mathcal{A}}$: ${p'_1}^{\circ}\mathcal{C} \subseteq \mathcal{D}\}$ is finite

provided the parameters meet the following conditions:

- $\mathcal{C}$ is finite,
- $\mathcal{D}$ is finite, and
- For all elements p_1, p_2 of $\mathcal{B}^{\mathcal{A}}$ such that $p_1 \restriction \mathcal{C} = p_2 \restriction \mathcal{C}$ holds $\mathcal{F}(p_1) = \mathcal{F}(p_2)$.

The scheme *FunctChoice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathrm{Fin}\,\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f_1 from $\mathcal{A}$ into $\mathcal{B}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, f_1(t)]$

provided the following requirement is met:

- For every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element f_1 of $\mathcal{B}$ such that $\mathcal{P}[t, f_1]$.

The scheme *FuncsChoice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathrm{Fin}\,\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists an element f_1 of $\mathcal{B}^{\mathcal{A}}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, f_1(t)]$

provided the parameters meet the following requirement:

- For every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element f_1 of $\mathcal{B}$ such that $\mathcal{P}[t, f_1]$.

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