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Filters - Part II. Quotient Lattices Modulo Filters and Direct Product of Two Lattices

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Summary. Binary and unary operation preserving binary relations and quotients of those operations modulo equivalence relations are introduced. It is shown that the quotients inherit some important properties (commutativity, associativity, distributivity, etc.). Based on it, the quotient (also called factor) lattice modulo a filter (i.e. modulo the equivalence relation w.r.t the filter) is introduced. Similarly, some properties of the direct product of two binary (unary) operations are present and then the direct product of two lattices is introduced. Besides, the heredity of distributivity, modularity, completeness, etc., for the product of lattices is also shown. Finally, the concept of isomorphic lattices is introduced, and there is shown that every Boolean lattice B is isomorphic with the direct product of the factor lattice $B/[a]$ and the lattice $\mathrm{latt}[a]$, where a is an element of B.

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The articles [10], [6], [12], [13], [4], [5], [3], [8], [9], [1], [7], [14], [2], and [11] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: L, L_1, L_2 are lattices, F_1, F_2 are filters of L, p, q are elements of L, p_1, q_1 are elements of L_1, p_2, q_2 are elements of L_2, x, x_1, y, y_1 are sets, D, D_1, D_2 are non empty sets, R is a binary relation, R_1 is an equivalence relation of D, a, b, d are elements of D, a_1, b_1 are elements of D_1, a_2, b_2 are elements of D_2, B is a Boolean lattice, F_3 is a filter of B, I is an implicative lattice, F_4 is a filter of I, i, i_1, i_2, j, j_1, j_2, k are elements of I, f_1, g_1 are binary operations on D_1, and f_2, g_2 are binary operations on D_2.

The following two propositions are true:

(1) $F_1 \cap F_2$ is a filter of L.

(2) If $[p) = [q)$, then $p = q$.

Let us consider L, F_1, F_2. Then $F_1 \cap F_2$ is a filter of L.

Let us consider D, R. A unary operation on D is said to be a unary R-congruent operation on D if:

(Def. 1) For all elements x, y of D such that $\langle x, y \rangle \in R$ holds $\langle \mathrm{it}(x), \mathrm{it}(y) \rangle \in R$.

A binary operation on D is said to be a binary R-congruent operation on D if:

(Def. 2) For all elements x_1, y_1, x_2, y_2 of D such that $\langle x_1, y_1 \rangle \in R$ and $\langle x_2, y_2 \rangle \in R$ holds $\langle \mathrm{it}(x_1, x_2), \mathrm{it}(y_1, y_2) \rangle \in R$.

In the sequel F, G denote binary R_1-congruent operations on D.

Let us consider D and let R be an equivalence relation of D. A unary operation on R is a unary R-congruent operation on D. A binary operation on R is a binary R-congruent operation on D. Note that $\mathrm{Classes}\,R$ is non empty.

Let us consider D, let R be an equivalence relation of D, and let d be an element of D. Then $[d]_R$ is an element of $\mathrm{Classes}\,R$.

Let us consider D, let R be an equivalence relation of D, and let u be a unary operation on D. Let us assume that u is a unary R-congruent operation on D. The functor $u_{/R}$ yields a unary operation on $\mathrm{Classes}\,R$ and is defined by:

(Def. 3) For all x, y such that $x \in \mathrm{Classes}\,R$ and $y \in x$ holds $u_{/R}(x) = [u(y)]_R$.

Let us consider D, let R be an equivalence relation of D, and let b be a binary operation on D. Let us assume that b is a binary R-congruent operation on D. The functor $b_{/R}$ yielding a binary operation on $\mathrm{Classes}\,R$ is defined as follows:

(Def. 4) For all x, y, x_1, y_1 such that $x \in \mathrm{Classes}\,R$ and $y \in \mathrm{Classes}\,R$ and $x_1 \in x$ and $y_1 \in y$ holds $b_{/R}(x, y) = [b(x_1, y_1)]_R$.

We now state a number of propositions:

(3) $F_{/R_1}([a]_{(R_1)}, [b]_{(R_1)}) = [F(a, b)]_{(R_1)}$.

(4) If F is commutative, then $F_{/R_1}$ is commutative.

(5) If F is associative, then $F_{/R_1}$ is associative.

(6) If d is a left unity w.r.t. F, then $[d]_{(R_1)}$ is a left unity w.r.t. $F_{/R_1}$.

(7) If d is a right unity w.r.t. F, then $[d]_{(R_1)}$ is a right unity w.r.t. $F_{/R_1}$.

(8) If d is a unity w.r.t. F, then $[d]_{(R_1)}$ is a unity w.r.t. $F_{/R_1}$.

(9) If F is left distributive w.r.t. G, then $F_{/R_1}$ is left distributive w.r.t. $G_{/R_1}$.

(10) If F is right distributive w.r.t. G, then $F_{/R_1}$ is right distributive w.r.t. $G_{/R_1}$.

(11) If F is distributive w.r.t. G, then $F_{/R_1}$ is distributive w.r.t. $G_{/R_1}$.

(12) If F absorbs G, then $F_{/R_1}$ absorbs $G_{/R_1}$.

(13) The join operation of I is a binary $\equiv_{(F_4)}$-congruent operation on the carrier of I.

(14) The meet operation of I is a binary $\equiv_{(F_4)}$-congruent operation on the carrier of I.

Let L be a lattice and let F be a filter of L. Let us assume that L is an implicative lattice. The functor $L_{/F}$ yielding a strict lattice is defined by the condition (Def. 5).

(Def. 5) Let R be an equivalence relation of the carrier of L. Suppose $R = \equiv_F$. Then $L_{/F} = \langle \mathrm{Classes}\,R, (\text{the join operation of } L)_{/R}, (\text{the meet operation of } L)_{/R} \rangle$.

Let L be a lattice, let F be a filter of L, and let a be an element of L. Let us assume that L is an implicative lattice. The functor $a_{/F}$ yields an element of $L_{/F}$ and is defined as follows:

(Def. 6) For every equivalence relation R of the carrier of L such that $R = \equiv_F$ holds $a_{/F} = [a]_R$.

We now state several propositions:

(15) $i_{/F_4} \sqcup j_{/F_4} = (i \sqcup j)_{/F_4}$ and $i_{/F_4} \sqcap j_{/F_4} = (i \sqcap j)_{/F_4}$.

(16) $i_{/F_4} \sqsubseteq j_{/F_4}$ iff $i \Rightarrow j \in F_4$.

(17) $i \sqcap j \Rightarrow k = i \Rightarrow (j \Rightarrow k)$.

(18) If I is lower-bounded, then $I_{/F_4}$ is a lower bound lattice and $\bot_{I_{/F_4}} = (\bot_I)_{/F_4}$.

(19) $I_{/F_4}$ is an upper bound lattice and $\top_{I_{/F_4}} = (\top_I)_{/F_4}$.

(20) $I_{/F_4}$ is implicative.

(21) $B_{/F_3}$ is a Boolean lattice.

Let D_1, D_2 be sets, let f_1 be a binary operation on D_1, and let f_2 be a binary operation on D_2. Then $|{:}f_1, f_2{:}|$ is a binary operation on $[{:}D_1, D_2{:}]$.

Next we state a number of propositions:

(22) $|{:}f_1, f_2{:}|(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) = \langle f_1(a_1, b_1), f_2(a_2, b_2)\rangle$.

(23) f_1 is commutative and f_2 is commutative iff $|{:}f_1, f_2{:}|$ is commutative.

(24) f_1 is associative and f_2 is associative iff $|{:}f_1, f_2{:}|$ is associative.

(25) a_1 is a left unity w.r.t. f_1 and a_2 is a left unity w.r.t. f_2 iff $\langle a_1, a_2\rangle$ is a left unity w.r.t. $|{:}f_1, f_2{:}|$.

(26) a_1 is a right unity w.r.t. f_1 and a_2 is a right unity w.r.t. f_2 iff $\langle a_1, a_2\rangle$ is a right unity w.r.t. $|{:}f_1, f_2{:}|$.

(27) a_1 is a unity w.r.t. f_1 and a_2 is a unity w.r.t. f_2 iff $\langle a_1, a_2\rangle$ is a unity w.r.t. $|{:}f_1, f_2{:}|$.

(28) f_1 is left distributive w.r.t. g_1 and f_2 is left distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is left distributive w.r.t. $|{:}g_1, g_2{:}|$.

(29) f_1 is right distributive w.r.t. g_1 and f_2 is right distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is right distributive w.r.t. $|{:}g_1, g_2{:}|$.

(30) f_1 is distributive w.r.t. g_1 and f_2 is distributive w.r.t. g_2 iff $|{:}f_1, f_2{:}|$ is distributive w.r.t. $|{:}g_1, g_2{:}|$.

(31) f_1 absorbs g_1 and f_2 absorbs g_2 iff $|{:}f_1, f_2{:}|$ absorbs $|{:}g_1, g_2{:}|$.

Let L_1, L_2 be non empty lattice structures. The functor $[{:}L_1, L_2{:}]$ yields a strict lattice structure and is defined by the condition (Def. 7).

(Def. 7) $[{:}L_1, L_2{:}] = \langle [{:}\text{the carrier of } L_1, \text{the carrier of } L_2{:}], |{:}\text{the join operation of } L_1, \text{the join operation of } L_2{:}|, |{:}\text{the meet operation of } L_1, \text{the meet operation of } L_2{:}|\rangle$.

Let L_1, L_2 be non empty lattice structures. One can check that $[{:}L_1, L_2{:}]$ is non empty.

Let L be a lattice. The functor $\mathrm{LattRel}(L)$ yields a binary relation and is defined by:

(Def. 8) $\mathrm{LattRel}(L) = \{\langle p, q\rangle; p$ ranges over elements of L, q ranges over elements of L: $p \sqsubseteq q\}$.

We now state two propositions:

(32) $\langle p, q\rangle \in \mathrm{LattRel}(L)$ iff $p \sqsubseteq q$.

(33) $\mathrm{dom}\,\mathrm{LattRel}(L) =$ the carrier of L and $\mathrm{rng}\,\mathrm{LattRel}(L) =$ the carrier of L and $\mathrm{field}\,\mathrm{LattRel}(L) =$ the carrier of L.

Let L_1, L_2 be lattices. We say that L_1 and L_2 are isomorphic if and only if:

(Def. 9) $\mathrm{LattRel}(L_1)$ and $\mathrm{LattRel}(L_2)$ are isomorphic.

Let us notice that the predicate L_1 and L_2 are isomorphic is reflexive and symmetric. One can verify that $[{:}L_1, L_2{:}]$ is lattice-like.

Next we state two propositions:

(34) Let L_1, L_2, L_3 be lattices. Suppose L_1 and L_2 are isomorphic and L_2 and L_3 are isomorphic. Then L_1 and L_3 are isomorphic.

(35) For all non empty lattice structures L_1, L_2 such that $[:L_1, L_2:]$ is a lattice holds L_1 is a lattice and L_2 is a lattice.

Let L_1, L_2 be lattices, let a be an element of L_1, and let b be an element of L_2. Then $\langle a, b\rangle$ is an element of $[:L_1, L_2:]$.

Next we state a number of propositions:

(36) $\langle p_1, p_2\rangle \sqcup \langle q_1, q_2\rangle = \langle p_1 \sqcup q_1, p_2 \sqcup q_2\rangle$ and $\langle p_1, p_2\rangle \sqcap \langle q_1, q_2\rangle = \langle p_1 \sqcap q_1, p_2 \sqcap q_2\rangle$.

(37) $\langle p_1, p_2\rangle \sqsubseteq \langle q_1, q_2\rangle$ iff $p_1 \sqsubseteq q_1$ and $p_2 \sqsubseteq q_2$.

(38) L_1 is modular and L_2 is modular iff $[:L_1, L_2:]$ is modular.

(39) L_1 is a distributive lattice and L_2 is a distributive lattice if and only if $[:L_1, L_2:]$ is a distributive lattice.

(40) L_1 is lower-bounded and L_2 is lower-bounded iff $[:L_1, L_2:]$ is lower-bounded.

(41) L_1 is upper-bounded and L_2 is upper-bounded iff $[:L_1, L_2:]$ is upper-bounded.

(42) L_1 is bounded and L_2 is bounded iff $[:L_1, L_2:]$ is bounded.

(43) If L_1 is a lower bound lattice and L_2 is a lower bound lattice, then $\perp_{[:L_1,L_2:]} = \langle \perp_{(L_1)}, \perp_{(L_2)}\rangle$.

(44) If L_1 is an upper bound lattice and L_2 is an upper bound lattice, then $\top_{[:L_1,L_2:]} = \langle \top_{(L_1)}, \top_{(L_2)}\rangle$.

(45) Suppose L_1 is a bound lattice and L_2 is a bound lattice. Then p_1 is a complement of q_1 and p_2 is a complement of q_2 if and only if $\langle p_1, p_2\rangle$ is a complement of $\langle q_1, q_2\rangle$.

(46) L_1 is a complemented lattice and L_2 is a complemented lattice if and only if $[:L_1, L_2:]$ is a complemented lattice.

(47) L_1 is a Boolean lattice and L_2 is a Boolean lattice iff $[:L_1, L_2:]$ is a Boolean lattice.

(48) L_1 is implicative and L_2 is implicative iff $[:L_1, L_2:]$ is implicative.

(49) $[:L_1, L_2:]^\circ = [:{L_1}^\circ, {L_2}^\circ:]$.

(50) $[:L_1, L_2:]$ and $[:L_2, L_1:]$ are isomorphic.

We follow the rules: B is a Boolean lattice and a, b, c, d are elements of B.

Next we state a number of propositions:

(51) $a \Leftrightarrow b = (a \sqcap b) \sqcup (a^{\rm c} \sqcap b^{\rm c})$.

(52) $(a \Rightarrow b)^{\rm c} = a \sqcap b^{\rm c}$ and $(a \Leftrightarrow b)^{\rm c} = (a \sqcap b^{\rm c}) \sqcup (a^{\rm c} \sqcap b)$ and $(a \Leftrightarrow b)^{\rm c} = a \Leftrightarrow b^{\rm c}$ and $(a \Leftrightarrow b)^{\rm c} = a^{\rm c} \Leftrightarrow b$.

(53) If $a \Leftrightarrow b = a \Leftrightarrow c$, then $b = c$.

(54) $a \Leftrightarrow (a \Leftrightarrow b) = b$.

(55) $i \sqcup j \Rightarrow i = j \Rightarrow i$ and $i \Rightarrow i \sqcap j = i \Rightarrow j$.

(56) $i \Rightarrow j \sqsubseteq i \Rightarrow j \sqcup k$ and $i \Rightarrow j \sqsubseteq i \sqcap k \Rightarrow j$ and $i \Rightarrow j \sqsubseteq i \Rightarrow k \sqcup j$ and $i \Rightarrow j \sqsubseteq k \sqcap i \Rightarrow j$.

(57) $(i \Rightarrow k) \sqcap (j \Rightarrow k) \sqsubseteq i \sqcup j \Rightarrow k$.

(58) $(i \Rightarrow j) \sqcap (i \Rightarrow k) \sqsubseteq i \Rightarrow j \sqcap k$.

(59) If $i_1 \Leftrightarrow i_2 \in F_4$ and $j_1 \Leftrightarrow j_2 \in F_4$, then $i_1 \sqcup j_1 \Leftrightarrow i_2 \sqcup j_2 \in F_4$ and $i_1 \sqcap j_1 \Leftrightarrow i_2 \sqcap j_2 \in F_4$.

(60) If $i \in [k]_{\equiv_{(F_4)}}$ and $j \in [k]_{\equiv_{(F_4)}}$, then $i \sqcup j \in [k]_{\equiv_{(F_4)}}$ and $i \sqcap j \in [k]_{\equiv_{(F_4)}}$.

(61) $c \sqcup (c \Leftrightarrow d) \in [c]_{\equiv_{[d)}}$ and for every b such that $b \in [c]_{\equiv_{[d)}}$ holds $b \sqsubseteq c \sqcup (c \Leftrightarrow d)$.

(62) B and $[: B_{/[a)}, \mathbb{L}_{[a)} :]$ are isomorphic.

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