JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Filters — Part I

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Summary. Filters of a lattice, maximal filters (ultrafilters), operation to create a filter generating by an element or by a non-empty subset of the lattice are discussed. Besides, there are introduced implicative lattices such that for every two elements there is an element being pseudo-complement of them. Some facts concerning these concepts are presented too, i.e. for any proper filter there exists an ultrafilter consisting it.

MML Identifier: FILTER_0.

WWW: http://mizar.org/JFM/Vol2/filter_0.html

The articles [7], [4], [5], [1], [8], [11], [3], [2], [12], [6], [9], and [10] provide the notation and terminology for this paper.

We adopt the following convention: L is a lattice, p, p_1, q, q_1, r are elements of L, and x is a set.

We now state several propositions:

(1) If $p \sqsubseteq q$, then $r \sqcup p \sqsubseteq r \sqcup q$ and $p \sqcup r \sqsubseteq q \sqcup r$ and $p \sqcup r \sqsubseteq r \sqcup q$ and $r \sqcup p \sqsubseteq q \sqcup r$.

(2) If $p \sqsubseteq r$, then $p \sqcap q \sqsubseteq r$ and $q \sqcap p \sqsubseteq r$.

(3) If $p \sqsubseteq r$, then $p \sqsubseteq q \sqcup r$ and $p \sqsubseteq r \sqcup q$.

(4) If $p \sqsubseteq p_1$ and $q \sqsubseteq q_1$, then $p \sqcup q \sqsubseteq p_1 \sqcup q_1$ and $p \sqcup q \sqsubseteq q_1 \sqcup p_1$.

(5) If $p \sqsubseteq p_1$ and $q \sqsubseteq q_1$, then $p \sqcap q \sqsubseteq p_1 \sqcap q_1$ and $p \sqcap q \sqsubseteq q_1 \sqcap p_1$.

(6) If $p \sqsubseteq r$ and $q \sqsubseteq r$, then $p \sqcup q \sqsubseteq r$.

(7) If $r \sqsubseteq p$ and $r \sqsubseteq q$, then $r \sqsubseteq p \sqcap q$.

Let us consider L. A non empty subset of L is said to be a filter of L if:

(Def. 1) $p \in$ it and $q \in$ it iff $p \sqcap q \in$ it.

We now state the proposition

(9)[1] Let D be a non empty subset of L. Then D is a filter of L if and only if the following conditions are satisfied:

(i) for all p, q such that $p \in D$ and $q \in D$ holds $p \sqcap q \in D$, and

(ii) for all p, q such that $p \in D$ and $p \sqsubseteq q$ holds $q \in D$.

In the sequel H, F denote filters of L.

One can prove the following propositions:

[1] The proposition (8) has been removed.

(10) If $p \in H$, then $p \sqcup q \in H$ and $q \sqcup p \in H$.

(11) There exists p such that $p \in H$.

(12) If L is an upper bound lattice, then $\top_L \in H$.

(13) If L is an upper bound lattice, then $\{\top_L\}$ is a filter of L.

(14) If $\{p\}$ is a filter of L, then L is upper-bounded.

(15) The carrier of L is a filter of L.

Let us consider L. The functor $[L)$ yields a filter of L and is defined by:

(Def. 2) $[L) =$ the carrier of L.

Let us consider L, p. The functor $[p)$ yields a filter of L and is defined by:

(Def. 3) $[p) = \{q : p \sqsubseteq q\}$.

One can prove the following propositions:

(18)[2] $q \in [p)$ iff $p \sqsubseteq q$.

(19) $p \in [p)$ and $p \sqcup q \in [p)$ and $q \sqcup p \in [p)$.

(20) If L is a lower bound lattice, then $[L) = [\bot_L)$.

Let us consider L, F. We say that F is an ultrafilter if and only if:

(Def. 4) $F \neq$ the carrier of L and for every H such that $F \subseteq H$ and $H \neq$ the carrier of L holds $F = H$.

We introduce F is an ultrafilter as a synonym of F is an ultrafilter.
The following three propositions are true:

(22)[3] If L is lower-bounded, then for every F such that $F \neq$ the carrier of L there exists H such that $F \subseteq H$ and H is an ultrafilter.

(23) If there exists r such that $p \sqcap r \neq p$, then $[p) \neq$ the carrier of L.

(24) If L is a lower bound lattice and $p \neq \bot_L$, then there exists H such that $p \in H$ and H is an ultrafilter.

In the sequel D denotes a non empty subset of L.
Let us consider L, D. The functor $[D)$ yielding a filter of L is defined as follows:

(Def. 5) $D \subseteq [D)$ and for every F such that $D \subseteq F$ holds $[D) \subseteq F$.

Next we state the proposition

(26)[4] $[F) = F$.

In the sequel D_1, D_2 are non empty subsets of L.
One can prove the following propositions:

(27) If $D_1 \subseteq D_2$, then $[D_1) \subseteq [D_2)$.

(29)[5] If $p \in D$, then $[p) \subseteq [D)$.

(30) If $D = \{p\}$, then $[D) = [p)$.

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[2] The propositions (16) and (17) have been removed.
[3] The proposition (21) has been removed.
[4] The proposition (25) has been removed.
[5] The proposition (28) has been removed.

(31) If L is a lower bound lattice and $\bot_L \in D$, then $[D) = [L)$ and $[D) =$ the carrier of L.

(32) If L is a lower bound lattice and $\bot_L \in F$, then $F = [L)$ and $F =$ the carrier of L.

Let us consider L, F. We say that F is prime if and only if:

(Def. 6) $p \sqcup q \in F$ iff $p \in F$ or $q \in F$.

Next we state the proposition

(34)[6] If L is a Boolean lattice, then for all p, q holds $p \sqcap (p^{\rm c} \sqcup q) \sqsubseteq q$ and for every r such that $p \sqcap r \sqsubseteq q$ holds $r \sqsubseteq p^{\rm c} \sqcup q$.

Let I_1 be a non empty lattice structure. We say that I_1 is implicative if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let p, q be elements of I_1. Then there exists an element r of I_1 such that $p \sqcap r \sqsubseteq q$ and for every element r_1 of I_1 such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

Let us observe that there exists a lattice which is strict and implicative.
An implicative lattice is an implicative lattice.
Let us consider L, p, q. Let us assume that L is an implicative lattice. The functor $p \Rightarrow q$ yields an element of L and is defined by:

(Def. 8) $p \sqcap (p \Rightarrow q) \sqsubseteq q$ and for every r such that $p \sqcap r \sqsubseteq q$ holds $r \sqsubseteq p \Rightarrow q$.

In the sequel I denotes an implicative lattice and i denotes an element of I.
One can prove the following propositions:

(37)[7] I is upper-bounded.

(38) $i \Rightarrow i = \top_I$.

(39) I is distributive.

In the sequel B is a Boolean lattice and F_1, H_1 are filters of B.
Next we state the proposition

(40) B is implicative.

Let us note that every lattice which is implicative is also distributive.
For simplicity, we follow the rules: I denotes an implicative lattice, i, j, k denote elements of I, D_3 denotes a non empty subset of I, and F_2 denotes a filter of I.
We now state two propositions:

(41) If $i \in F_2$ and $i \Rightarrow j \in F_2$, then $j \in F_2$.

(42) If $j \in F_2$, then $i \Rightarrow j \in F_2$.

Let us consider L, D_1, D_2. The functor $D_1 \sqcap D_2$ yielding a subset of L is defined as follows:

(Def. 9) $D_1 \sqcap D_2 = \{p \sqcap q : p \in D_1 \ \wedge \ q \in D_2\}$.

Let us consider L, D_1, D_2. Observe that $D_1 \sqcap D_2$ is non empty.
We now state three propositions:

(44)[8] If $p \in D_1$ and $q \in D_2$, then $p \sqcap q \in D_1 \sqcap D_2$ and $q \sqcap p \in D_1 \sqcap D_2$.

(45) If $x \in D_1 \sqcap D_2$, then there exist p, q such that $x = p \sqcap q$ and $p \in D_1$ and $q \in D_2$.

[6] The proposition (33) has been removed.
[7] The propositions (35) and (36) have been removed.
[8] The proposition (43) has been removed.

(46) $D_1 \sqcap D_2 = D_2 \sqcap D_1$.

Let L be a distributive lattice and let F_3, F_4 be filters of L. Then $F_3 \sqcap F_4$ is a filter of L.
Let L be a Boolean lattice and let F_3, F_4 be filters of L. Then $F_3 \sqcap F_4$ is a filter of L.
The following propositions are true:

(47) $[D_1 \cup D_2) = [[D_1) \cup D_2)$ and $[D_1 \cup D_2) = [D_1 \cup [D_2))$.

(48) $[F \cup H) = \{r : \bigvee_{p,q} (p \sqcap q \sqsubseteq r \ \wedge\ p \in F \ \wedge\ q \in H)\}$.

(49) $F \subseteq F \sqcap H$ and $H \subseteq F \sqcap H$.

(50) $[F \cup H) = [F \sqcap H)$.

In the sequel F_3, F_4 are filters of I.
Next we state four propositions:

(51) $[F_3 \cup F_4) = F_3 \sqcap F_4$.

(52) $[F_1 \cup H_1) = F_1 \sqcap H_1$.

(53) If $j \in [D_3 \cup \{i\})$, then $i \Rightarrow j \in [D_3)$.

(54) If $i \Rightarrow j \in F_2$ and $j \Rightarrow k \in F_2$, then $i \Rightarrow k \in F_2$.

In the sequel a, b, c denote elements of B.
One can prove the following propositions:

(55) $a \Rightarrow b = a^{\rm c} \sqcup b$.

(56) $a \sqsubseteq b$ iff $a \sqcap b^{\rm c} = \bot_B$.

(57) F_1 is an ultrafilter iff $F_1 \neq$ the carrier of B and for every a holds $a \in F_1$ or $a^{\rm c} \in F_1$.

(58) $F_1 \neq [B)$ and F_1 is prime iff F_1 is an ultrafilter.

(59) If F_1 is an ultrafilter, then for every a holds $a \in F_1$ iff $a^{\rm c} \notin F_1$.

(60) If $a \neq b$, then there exists F_1 such that F_1 is an ultrafilter but $a \in F_1$ and $b \notin F_1$ or $a \notin F_1$ and $b \in F_1$.

In the sequel o_1, o_2 denote binary operations on F.
Let us consider L, F. The functor $\mathbb{L}_F$ yielding a lattice is defined by the condition (Def. 10).

(Def. 10) There exist o_1, o_2 such that $o_1 =$ (the join operation of L)$\restriction$($[: F, F :]$ **qua** set) and $o_2 =$ (the meet operation of L)$\restriction$($[: F, F :]$ **qua** set) and $\mathbb{L}_F = \langle F, o_1, o_2 \rangle$.

Let us consider L, F. Note that $\mathbb{L}_F$ is strict.
Next we state a number of propositions:

(62)[9] For every strict lattice L holds $\mathbb{L}_{[L)} = L$.

(63)(i) The carrier of $\mathbb{L}_F = F$,

(ii) the join operation of $\mathbb{L}_F =$ (the join operation of L)$\restriction$($[: F, F :]$ **qua** set), and

(iii) the meet operation of $\mathbb{L}_F =$ (the meet operation of L)$\restriction$($[: F, F :]$ **qua** set).

(64) For all elements p', q' of $\mathbb{L}_F$ such that $p = p'$ and $q = q'$ holds $p \sqcup q = p' \sqcup q'$ and $p \sqcap q = p' \sqcap q'$.

(65) For all elements p', q' of $\mathbb{L}_F$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \sqsubseteq q'$.

[9] The proposition (61) has been removed.

(66) If L is upper-bounded, then $\mathbb{L}_F$ is upper-bounded.

(67) If L is modular, then $\mathbb{L}_F$ is modular.

(68) If L is distributive, then $\mathbb{L}_F$ is distributive.

(69) If L is an implicative lattice, then $\mathbb{L}_F$ is implicative.

(70) $\mathbb{L}_{[p)}$ is lower-bounded.

(71) $\perp_{\mathbb{L}_{[p)}} = p$.

(72) If L is upper-bounded, then $\top_{\mathbb{L}_{[p)}} = \top_L$.

(73) If L is an upper bound lattice, then $\mathbb{L}_{[p)}$ is bounded.

(74) If L is a complemented lattice and a modular lattice, then $\mathbb{L}_{[p)}$ is a complemented lattice.

(75) If L is a Boolean lattice, then $\mathbb{L}_{[p)}$ is a Boolean lattice.

Let us consider L, p, q. The functor $p \Leftrightarrow q$ yielding an element of L is defined by:

(Def. 11) $p \Leftrightarrow q = (p \Rightarrow q) \sqcap (q \Rightarrow p)$.

One can prove the following two propositions:

(77)[10] $p \Leftrightarrow q = q \Leftrightarrow p$.

(78) If $i \Leftrightarrow j \in F_2$ and $j \Leftrightarrow k \in F_2$, then $i \Leftrightarrow k \in F_2$.

Let us consider L, F. The functor $\equiv_F$ yielding a binary relation is defined as follows:

(Def. 12) $\operatorname{field}(\equiv_F) \subseteq$ the carrier of L and for all p, q holds $\langle p, q \rangle \in \equiv_F$ iff $p \Leftrightarrow q \in F$.

The following propositions are true:

(80)[11] $\equiv_F$ is a binary relation on the carrier of L.

(81) If L is an implicative lattice, then $\equiv_F$ is reflexive in the carrier of L.

(82) $\equiv_F$ is symmetric in the carrier of L.

(83) If L is an implicative lattice, then $\equiv_F$ is transitive in the carrier of L.

(84) If L is an implicative lattice, then $\equiv_F$ is an equivalence relation of the carrier of L.

(85) If L is an implicative lattice, then $\operatorname{field}(\equiv_F) =$ the carrier of L.

Let us consider I, F_2. Then $\equiv_{(F_2)}$ is an equivalence relation of the carrier of I.
Let us consider B, F_1. Then $\equiv_{(F_1)}$ is an equivalence relation of the carrier of B.
Let us consider L, F, p, q. The predicate $p \equiv_F q$ is defined as follows:

(Def. 13) $p \Leftrightarrow q \in F$.

Next we state four propositions:

(87)[12] $p \equiv_F q$ iff $\langle p, q \rangle \in \equiv_F$.

(88) $i \equiv_{F_2} i$ and $a \equiv_{F_1} a$.

(89) If $p \equiv_F q$, then $q \equiv_F p$.

(90) If $i \equiv_{F_2} j$ and $j \equiv_{F_2} k$, then $i \equiv_{F_2} k$ and if $a \equiv_{F_1} b$ and $b \equiv_{F_1} c$, then $a \equiv_{F_1} c$.

[10] The proposition (76) has been removed.
[11] The proposition (79) has been removed.
[12] The proposition (86) has been removed.

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Received July 3, 1990

Published January 2, 2004