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Real Function Uniform Continuity[1]

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Summary. The uniform continuity for real functions is introduced. More theorems concerning continuous functions are given. (See [10]) The Darboux Theorem is exposed. Algebraic features for uniformly continuous functions are presented. Various facts, e.g., a continuous function on a compact set is uniformly continuous are proved.

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The articles [12], [14], [1], [13], [3], [2], [9], [15], [5], [4], [6], [7], [8], [11], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: X, X_1, Z, Z_1 denote sets, s, g, r, p, x_1, x_2 denote real numbers, Y denotes a subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Let us consider f, X. We say that f is uniformly continuous on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $|x_1 - x_2| < s$ holds $|f(x_1) - f(x_2)| < r$.

One can prove the following propositions:

(2)[1] If f is uniformly continuous on X and $X_1 \subseteq X$, then f is uniformly continuous on X_1.

(3) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(4) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(5) If f is uniformly continuous on X, then $p\,f$ is uniformly continuous on X.

(6) If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(7) If f is uniformly continuous on X, then $|f|$ is uniformly continuous on X.

(8) Suppose f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1 and f_1 is bounded on Z and f_2 is bounded on Z_1. Then $f_1\ f_2$ is uniformly continuous on $X \cap Z \cap X_1 \cap Z_1$.

(9) If f is uniformly continuous on X, then f is continuous on X.

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[1] The proposition (1) has been removed.

(10) If f is Lipschitzian on X, then f is uniformly continuous on X.

(11) For all f, Y such that Y is compact and f is continuous on Y holds f is uniformly continuous on Y.

(13)[2] If $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is uniformly continuous on Y, then $f^{\circ}Y$ is compact.

(14) Let given f, Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is uniformly continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f^{\circ}Y)$ and $f(x_2) = \inf(f^{\circ}Y)$.

(15) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is uniformly continuous on X.

(16) If $p \leq g$ and f is continuous on $[p,g]$, then for every r such that $r \in [f(p), f(g)] \cup [f(g), f(p)]$ there exists s such that $s \in [p,g]$ and $r = f(s)$.

(17) If $p \leq g$ and f is continuous on $[p,g]$, then for every r such that $r \in [\inf(f^{\circ}[p,g]), \sup(f^{\circ}[p,g])]$ there exists s such that $s \in [p,g]$ and $r = f(s)$.

(18) If f is one-to-one and $p \leq g$ and f is continuous on $[p,g]$, then f is increasing on $[p,g]$ and decreasing on $[p,g]$.

(19) Suppose f is one-to-one and $p \leq g$ and f is continuous on $[p,g]$. Then $\inf(f^{\circ}[p,g]) = f(p)$ and $\sup(f^{\circ}[p,g]) = f(g)$ or $\inf(f^{\circ}[p,g]) = f(g)$ and $\sup(f^{\circ}[p,g]) = f(p)$.

(20) If $p \leq g$ and f is continuous on $[p,g]$, then $f^{\circ}[p,g] = [\inf(f^{\circ}[p,g]), \sup(f^{\circ}[p,g])]$.

(21) Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. If $p \leq g$ and f is continuous on $[p,g]$, then f^{-1} is continuous on $[\inf(f^{\circ}[p,g]), \sup(f^{\circ}[p,g])]$.

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[2] The proposition (12) has been removed.

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