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Real Function Continuity[1]

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Summary. The continuity of real functions is discussed. There is a function defined on some domain in real numbers which is continuous in a single point and on a subset of domain of the function. Main properties of real continuous functions are proved. Among them there is the Weierstraß Theorem. Algebraic features for real continuous functions are shown. Lipschitzian functions are introduced. The Lipschitz condition entails continuity.

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The articles [14], [17], [1], [15], [5], [2], [18], [4], [3], [12], [8], [7], [6], [16], [9], [10], [11], and [13] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: n denotes a natural number, X, X_1, Z, Z_1 denote sets, s, g, r, p, x_0, x_1, x_2 denote real numbers, s_1 denotes a sequence of real numbers, Y denotes a subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Let us consider f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 1) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

The following propositions are true:

(2)[1] f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f \cdot s_1$ is convergent and $f(x_0) = \lim(f \cdot s_1)$.

(3) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f(x_1) - f(x_0)| < r$.

(4) Let given f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_1 of $f(x_0)$ there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f(x_1) \in N_1$.

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[1] The proposition (1) has been removed.

(5) Let given f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_1 of $f(x_0)$ there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_1$.

(6) If $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(7) Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0 and $f_1\ f_2$ is continuous in x_0.

(8) If f is continuous in x_0, then $r\ f$ is continuous in x_0.

(9) If f is continuous in x_0, then $|f|$ is continuous in x_0 and $-f$ is continuous in x_0.

(10) If f is continuous in x_0 and $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous in x_0.

(11) If f_1 is continuous in x_0 and $f_1(x_0) \neq 0$ and f_2 is continuous in x_0, then $\frac{f_2}{f_1}$ is continuous in x_0.

(12) If f_1 is continuous in x_0 and f_2 is continuous in $f_1(x_0)$, then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider f, X. We say that f is continuous on X if and only if:

(Def. 2) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

The following propositions are true:

(14)[2] Let given X, f. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f(\lim s_1) = \lim(f \cdot s_1)$.

(15) f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f(x_1) - f(x_0)| < r$.

(16) f is continuous on X iff $f{\restriction}X$ is continuous on X.

(17) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(18) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(19) Let given X, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X and $f_1\ f_2$ is continuous on X.

(20) Let given X, X_1, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$ and $f_1\ f_2$ is continuous on $X \cap X_1$.

(21) For all r, X, f such that f is continuous on X holds $r\ f$ is continuous on X.

(22) If f is continuous on X, then $|f|$ is continuous on X and $-f$ is continuous on X.

(23) If f is continuous on X and $f^{-1}(\{0\}) = \emptyset$, then $\frac{1}{f}$ is continuous on X.

[2] The proposition (13) has been removed.

(24) If f is continuous on X and $(f\restriction X)^{-1}(\{0\}) = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(25) If f_1 is continuous on X and $f_1{}^{-1}(\{0\}) = \emptyset$ and f_2 is continuous on X, then $\frac{f_2}{f_1}$ is continuous on X.

(26) If f_1 is continuous on X and f_2 is continuous on $f_1{}^\circ X$, then $f_2 \cdot f_1$ is continuous on X.

(27) If f_1 is continuous on X and f_2 is continuous on X_1, then $f_2 \cdot f_1$ is continuous on $X \cap f_1{}^{-1}(X_1)$.

(28) If f is total and for all x_1, x_2 holds $f(x_1 + x_2) = f(x_1) + f(x_2)$ and there exists x_0 such that f is continuous in x_0, then f is continuous on $\mathbb{R}$.

(29) For every f such that $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$ holds $\mathrm{rng}\, f$ is compact.

(30) If $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(31) Let given f. Suppose $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$. Then there exist x_1, x_2 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = \sup \mathrm{rng}\, f$ and $f(x_2) = \inf \mathrm{rng}\, f$.

(32) Let given f, Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f^\circ Y)$ and $f(x_2) = \inf(f^\circ Y)$.

Let us consider f, X. We say that f is Lipschitzian on X if and only if:

(Def. 3) $X \subseteq \mathrm{dom}\, f$ and there exists r such that $0 < r$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ holds $|f(x_1) - f(x_2)| \leq r \cdot |x_1 - x_2|$.

We now state a number of propositions:

(34)[3] If f is Lipschitzian on X and $X_1 \subseteq X$, then f is Lipschitzian on X_1.

(35) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(36) If f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1, then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(37) Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1 and f_1 is bounded on Z and f_2 is bounded on Z_1. Then $f_1\ f_2$ is Lipschitzian on $X \cap Z \cap X_1 \cap Z_1$.

(38) If f is Lipschitzian on X, then $p\ f$ is Lipschitzian on X.

(39) If f is Lipschitzian on X, then $-f$ is Lipschitzian on X and $|f|$ is Lipschitzian on X.

(40) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is Lipschitzian on X.

(41) id_Y is Lipschitzian on Y.

(42) If f is Lipschitzian on X, then f is continuous on X.

(43) For every f such that there exists r such that $\mathrm{rng}\, f = \{r\}$ holds f is continuous on $\mathrm{dom}\, f$.

(44) If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is continuous on X.

(45) For every f such that for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = x_0$ holds f is continuous on $\mathrm{dom}\, f$.

(46) If $f = \mathrm{id}_{\mathrm{dom}\, f}$, then f is continuous on $\mathrm{dom}\, f$.

[3] The proposition (33) has been removed.

(47) If $Y \subseteq \mathrm{dom}\, f$ and $f \restriction Y = \mathrm{id}_Y$, then f is continuous on Y.

(48) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = r \cdot x_0 + p$, then f is continuous on X.

(49) If for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = {x_0}^{\mathbf{2}}$, then f is continuous on $\mathrm{dom}\, f$.

(50) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = {x_0}^{\mathbf{2}}$, then f is continuous on X.

(51) If for every x_0 such that $x_0 \in \mathrm{dom}\, f$ holds $f(x_0) = |x_0|$, then f is continuous on $\mathrm{dom}\, f$.

(52) If $X \subseteq \mathrm{dom}\, f$ and for every x_0 such that $x_0 \in X$ holds $f(x_0) = |x_0|$, then f is continuous on X.

(53) If $X \subseteq \mathrm{dom}\, f$ and f is monotone on X and there exist p, g such that $p \leq g$ and $f^\circ X = [p,g]$, then f is continuous on X.

(54) Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $p \leq g$ and $[p,g] \subseteq \mathrm{dom}\, f$ and f is increasing on $[p,g]$ and decreasing on $[p,g]$. Then $(f \restriction [p,g])^{-1}$ is continuous on $f^\circ[p,g]$.

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