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# Fundamental Types of Metric Affine Spaces

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**Summary.** We distinguish in the class of metric affine spaces some fundamental types of them. First we can assume the underlying affine space to satisfy classical affine configurational axiom; thus we come to Pappian, Desarguesian, Moufangian, and translation spaces. Next we distinguish the spaces satisfying theorem on three perpendiculars and the homogeneous spaces; these properties directly refer to some axioms involving orthogonality. Some known relationships between the introduced classes of structures are established. We also show that the commonly investigated models of metric affine geometry constructed in a real linear space with the help of a symmetric bilinear form belong to all the classes introduced in the paper.

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The articles [8], [1], [7], [3], [4], [2], [6], and [5] provide the notation and terminology for this paper.

Let $I_1$ be a metric affine space. We say that $I_1$ is Euclidean if and only if:

(Def. 1) For all elements $a$, $b$, $c$, $d$ of the carrier of $I_1$ such that $a,b \perp c,d$ and $b,c \perp a,d$ holds $b,d \perp a,c$.

Let $I_1$ be a metric affine space. We say that $I_1$ is Pappian if and only if:

(Def. 2) The affine reduct of $I_1$ is Pappian.

Let $I_1$ be a metric affine space. We say that $I_1$ is Desarguesian if and only if:

(Def. 3) The affine reduct of $I_1$ is Desarguesian.

Let $I_1$ be a metric affine space. We say that $I_1$ is Fanoian if and only if:

(Def. 4) The affine reduct of $I_1$ is Fanoian.

Let $I_1$ be a metric affine space. We say that $I_1$ is Moufangian if and only if:

(Def. 5) The affine reduct of $I_1$ is Moufangian.

Let $I_1$ be a metric affine space. We say that $I_1$ is translation if and only if:

(Def. 6) The affine reduct of $I_1$ is translational.

Let $I_1$ be a metric affine space. We say that $I_1$ is homogeneous if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let $o$, $a$, $a_1$, $b$, $b_1$, $c$, $c_1$ be elements of $I_1$. Suppose $o,a \perp o,a_1$ and $o,b \perp o,b_1$ and $o,c \perp o,c_1$ and $a,b \perp a_1,b_1$ and $a,c \perp a_1,c_1$ and $o,c \nparallel o,a$ and $o,a \nparallel o,b$. Then $b,c \perp b_1,c_1$.

In the sequel $M_1$ denotes a metric affine plane and $M_2$ denotes a metric affine space.
We now state a number of propositions:

(1) For all elements $a$, $b$, $c$ of $M_2$ such that not $\mathbf{L}(a,b,c)$ holds $a \neq b$ and $b \neq c$ and $a \neq c$.

(2) Let $a$, $b$, $c$, $d$ be elements of $M_1$ and $K$ be a subset of the carrier of $M_1$. If $a,b \perp K$ and $c,d \perp K$, then $a,b \upuparrows c,d$ and $a,b \upuparrows d,c$.

(3) Let $a$, $b$ be elements of $M_1$ and $A$, $K$ be subsets of the carrier of $M_1$. If $a \neq b$ and if $a,b \perp K$ or $b,a \perp K$ and if $a,b \perp A$ or $b,a \perp A$, then $K \mathbin{/\!/} A$.

(4) For all elements $x$, $y$, $z$ of $M_2$ such that $\mathbf{L}(x,y,z)$ holds $\mathbf{L}(x,z,y)$ and $\mathbf{L}(y,x,z)$ and $\mathbf{L}(y,z,x)$ and $\mathbf{L}(z,x,y)$ and $\mathbf{L}(z,y,x)$.

(5) For all elements $a$, $b$, $c$ of $M_1$ such that not $\mathbf{L}(a,b,c)$ there exists an element $d$ of $M_1$ such that $d,a \perp b,c$ and $d,b \perp a,c$.

(6) For all elements $a$, $b$, $c$, $d_1$, $d_2$ of $M_1$ such that not $\mathbf{L}(a,b,c)$ and $d_1,a \perp b,c$ and $d_1,b \perp a,c$ and $d_2,a \perp b,c$ and $d_2,b \perp a,c$ holds $d_1 = d_2$.

(7) For all elements $a$, $b$, $c$, $d$ of $M_1$ such that $a,b \perp c,d$ and $b,c \perp a,d$ and $\mathbf{L}(a,b,c)$ holds $a = c$ or $a = b$ or $b = c$.

(8) $M_1$ is Euclidean iff theorem on three perpendiculars holds in $M_1$.

(9) $M_1$ is homogeneous iff orthogonal version of Desargues Axiom holds in $M_1$.

(10) $M_1$ is Pappian iff Pappos Axiom holds in $M_1$.

(11) $M_1$ is Desarguesian iff Desargues Axiom holds in $M_1$.

(12) $M_1$ is Moufangian iff trapezium variant of Desargues Axiom holds in $M_1$.

(13) $M_1$ is translation iff minor Desargues Axiom holds in $M_1$.

(14) If $M_1$ is homogeneous, then $M_1$ is Desarguesian.

(15) If $M_1$ is Euclidean and Desarguesian, then $M_1$ is Pappian.

We adopt the following convention: $V$ is a real linear space and $w$, $y$, $u$, $v$ are vectors of $V$.
One can prove the following propositions:

(16) Let $o$, $c$, $c_1$, $a$, $a_1$, $a_2$ be elements of $M_1$. Suppose not $\mathbf{L}(o,c,a)$ and $o \neq c_1$ and $o,c \perp o,c_1$ and $o,a \perp o,a_1$ and $o,a \perp o,a_2$ and $c,a \perp c_1,a_1$ and $c,a \perp c_1,a_2$. Then $a_1 = a_2$.

(17) For all elements $o$, $c$, $c_1$, $a$ of $M_1$ such that not $\mathbf{L}(o,c,a)$ and $o \neq c_1$ there exists an element $a_1$ of $M_1$ such that $o,a \perp o,a_1$ and $c,a \perp c_1,a_1$.

(18) Let $a$, $b$ be real numbers. Suppose $w$, $y$ span the space and $0_V \neq u$ and $0_V \neq v$ and $u$, $v$ are orthogonal w.r.t. $w$, $y$ and $u = a \cdot w + b \cdot y$. Then there exists a real number $c$ such that $c \neq 0$ and $v = c \cdot b \cdot w + (-c \cdot a) \cdot y$.

(19) Suppose $w$, $y$ span the space and $0_V \neq u$ and $0_V \neq v$ and $u$, $v$ are orthogonal w.r.t. $w$, $y$. Then there exists a real number $c$ such that for all real numbers $a$, $b$ holds

(i) $a \cdot w + b \cdot y$, $c \cdot b \cdot w + (-c \cdot a) \cdot y$ are orthogonal w.r.t. $w$, $y$, and

(ii) $(a \cdot w + b \cdot y) - u$, $(c \cdot b \cdot w + (-c \cdot a) \cdot y) - v$ are orthogonal w.r.t. $w$, $y$.

(21)[1] If $w$, $y$ span the space and $M_1 = \mathbf{AMSp}(V,w,y)$, then LIN holds in $M_1$.

[1] The proposition (20) has been removed.

(22) Let $o$, $a$, $a_1$, $b$, $b_1$, $c$, $c_1$ be elements of $M_1$. Suppose $o,a \perp o,a_1$ and $o,b \perp o,b_1$ and $o,c \perp o,c_1$ and $a,b \perp a_1,b_1$ and $a,c \perp a_1,c_1$ and $o,c \nparallel o,a$ and $o,a \nparallel o,b$ and $o = a_1$. Then $b,c \perp b_1,c_1$.

(23) If $w$, $y$ span the space and $M_1 = \mathbf{AMSp}(V,w,y)$, then $M_1$ is homogeneous.

Let us observe that there exists a metric affine plane which is Euclidean, Pappian, Desarguesian, homogeneous, translation, Fanoian, and Moufangian.

Let us observe that there exists a metric affine space which is Euclidean, Pappian, Desarguesian, homogeneous, translation, Fanoian, and Moufangian.

The following proposition is true

(24) Suppose $w$, $y$ span the space and $M_1 = \mathbf{AMSp}(V,w,y)$. Then $M_1$ is an Euclidean Pappian Desarguesian homogeneous translation Fanoian Moufangian metric affine plane.

Let $M_1$ be a Pappian metric affine plane. Observe that the affine reduct of $M_1$ is Pappian.

Let $M_1$ be a Desarguesian metric affine plane. Note that the affine reduct of $M_1$ is Desarguesian.

Let $M_1$ be a Moufangian metric affine plane. Observe that the affine reduct of $M_1$ is Moufangian.

Let $M_1$ be a translation metric affine plane. Observe that the affine reduct of $M_1$ is translational.

Let $M_1$ be a Fanoian metric affine plane. Observe that the affine reduct of $M_1$ is Fanoian.

Let $M_1$ be a homogeneous metric affine plane. Note that the affine reduct of $M_1$ is Desarguesian.

Let $M_1$ be an Euclidean Desarguesian metric affine plane. One can check that the affine reduct of $M_1$ is Pappian.

Let $M_1$ be a Pappian metric affine space. Observe that the affine reduct of $M_1$ is Pappian.

Let $M_1$ be a Desarguesian metric affine space. One can verify that the affine reduct of $M_1$ is Desarguesian.

Let $M_1$ be a Moufangian metric affine space. Note that the affine reduct of $M_1$ is Moufangian.

Let $M_1$ be a translation metric affine space. Note that the affine reduct of $M_1$ is translational.

Let $M_1$ be a Fanoian metric affine space. Note that the affine reduct of $M_1$ is Fanoian.

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