JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

Angle and Triangle in Euclidian Topological Space

Akihiro Kubo
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

Summary. Two transformations between the complex space and 2-dimensional Euclidian topological space are defined. By them, the concept of argument is induced to 2-dimensional vectors using argument of complex number. Similarly, the concept of an angle is introduced using the angle of two complex numbers. The concept of a triangle and related concepts are also defined in *n*-dimensional Euclidian topological spaces.

MML Identifier: `EUCLID_3`.

WWW: `http://mizar.org/JFM/Vol15/euclid_3.html`

The articles [17], [20], [19], [21], [3], [13], [22], [4], [8], [18], [12], [5], [14], [16], [9], [2], [6], [7], [1], [11], [10], and [15] provide the notation and terminology for this paper.

We follow the rules: z, z_1, z_2 denote elements of $\mathbb{C}$, r, r_1, r_2, x_1, x_2 denote real numbers, and p, p_1, p_2, p_3, q denote points of $\mathcal{E}^2_{\rm T}$.

Let z be an element of $\mathbb{C}$. The functor cpx2euc(z) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 1) $\text{cpx2euc}(z) = [\Re(z), \Im(z)]$.

Let p be a point of $\mathcal{E}^2_{\rm T}$. The functor euc2cpx(p) yielding an element of $\mathbb{C}$ is defined by:

(Def. 2) $\text{euc2cpx}(p) = p_{\mathbf{1}} + p_{\mathbf{2}} i$.

Next we state a number of propositions:

(1) $\text{euc2cpx}(\text{cpx2euc}(z)) = z$.

(2) $\text{cpx2euc}(\text{euc2cpx}(p)) = p$.

(3) For every p there exists z such that $p = \text{cpx2euc}(z)$.

(4) For every z there exists p such that $z = \text{euc2cpx}(p)$.

(5) For all z_1, z_2 such that $\text{cpx2euc}(z_1) = \text{cpx2euc}(z_2)$ holds $z_1 = z_2$.

(6) For all p_1, p_2 such that $\text{euc2cpx}(p_1) = \text{euc2cpx}(p_2)$ holds $p_1 = p_2$.

(7) $(\text{cpx2euc}(z))_{\mathbf{1}} = \Re(z)$ and $(\text{cpx2euc}(z))_{\mathbf{2}} = \Im(z)$.

(8) $\Re(\text{euc2cpx}(p)) = p_{\mathbf{1}}$ and $\Im(\text{euc2cpx}(p)) = p_{\mathbf{2}}$.

(9) $\text{cpx2euc}(x_1 + x_2 i) = [x_1, x_2]$.

(10) $[\Re(z_1 + z_2), \Im(z_1 + z_2)] = [\Re(z_1) + \Re(z_2), \Im(z_1) + \Im(z_2)]$.

(11) $\text{cpx2euc}(z_1 + z_2) = \text{cpx2euc}(z_1) + \text{cpx2euc}(z_2)$.

(12) $(p_1+p_2)_{\mathbf{1}}+(p_1+p_2)_{\mathbf{2}}i=((p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}})+((p_1)_{\mathbf{2}}+(p_2)_{\mathbf{2}})i$.

(13) $\mathrm{euc2cpx}(p_1+p_2)=\mathrm{euc2cpx}(p_1)+\mathrm{euc2cpx}(p_2)$.

(14) $[\Re(-z),\Im(-z)]=[-\Re(z),-\Im(z)]$.

(15) $\mathrm{cpx2euc}(-z)=-\mathrm{cpx2euc}(z)$.

(16) $(-p)_{\mathbf{1}}+(-p)_{\mathbf{2}}i=-p_{\mathbf{1}}+(-p_{\mathbf{2}})i$.

(17) $\mathrm{euc2cpx}(-p)=-\mathrm{euc2cpx}(p)$.

(18) $\mathrm{cpx2euc}(z_1-z_2)=\mathrm{cpx2euc}(z_1)-\mathrm{cpx2euc}(z_2)$.

(19) $\mathrm{euc2cpx}(p_1-p_2)=\mathrm{euc2cpx}(p_1)-\mathrm{euc2cpx}(p_2)$.

(20) $\mathrm{cpx2euc}(0_{\mathbb{C}})=0_{\mathcal{E}^2_{\mathrm{T}}}$.

(21) $\mathrm{euc2cpx}(0_{\mathcal{E}^2_{\mathrm{T}}})=0_{\mathbb{C}}$.

(22) If $\mathrm{euc2cpx}(p)=0_{\mathbb{C}}$, then $p=0_{\mathcal{E}^2_{\mathrm{T}}}$.

(23) $\mathrm{cpx2euc}((r+0i)\cdot z)=r\cdot\mathrm{cpx2euc}(z)$.

(24) $(r+0i)\cdot(r_1+r_2i)=r\cdot r_1+(r\cdot r_2)i$.

(25) $\mathrm{euc2cpx}(r\cdot p)=(r+0i)\cdot\mathrm{euc2cpx}(p)$.

(26) $|\mathrm{euc2cpx}(p)|=\sqrt{(p_{\mathbf{1}})^2+(p_{\mathbf{2}})^2}$.

(27) For every finite sequence f of elements of $\mathbb{R}$ such that $\mathrm{len}\,f=2$ holds $|f|=\sqrt{f(1)^2+f(2)^2}$.

(28) For every finite sequence f of elements of $\mathbb{R}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{len}\,f=2$ and $p=f$ holds $|p|=|f|$.

(29) $|\mathrm{cpx2euc}(z)|=\sqrt{\Re(z)^2+\Im(z)^2}$.

(30) $|\mathrm{cpx2euc}(z)|=|z|$.

(31) $|\mathrm{euc2cpx}(p)|=|p|$.

Let us consider p. The functor $\mathrm{Arg}\,p$ yields a real number and is defined by:

(Def. 3) $\mathrm{Arg}\,p=\mathrm{Arg}\,\mathrm{euc2cpx}(p)$.

Next we state a number of propositions:

(32) For every element z of $\mathbb{C}$ and for every p such that $z=\mathrm{euc2cpx}(p)$ or $p=\mathrm{cpx2euc}(z)$ holds $\mathrm{Arg}\,z=\mathrm{Arg}\,p$.

(33) For every p holds $0\leq\mathrm{Arg}\,p$ and $\mathrm{Arg}\,p<2\cdot\pi$.

(34) For all real numbers x_1, x_2 and for every p such that $x_1=|p|\cdot\cos\mathrm{Arg}\,p$ and $x_2=|p|\cdot\sin\mathrm{Arg}\,p$ holds $p=[x_1,x_2]$.

(35) $\mathrm{Arg}(0_{\mathcal{E}^2_{\mathrm{T}}})=0$.

(36) For every p such that $p\neq0_{\mathcal{E}^2_{\mathrm{T}}}$ holds if $\mathrm{Arg}\,p<\pi$, then $\mathrm{Arg}(-p)=\mathrm{Arg}\,p+\pi$ and if $\mathrm{Arg}\,p\geq\pi$, then $\mathrm{Arg}(-p)=\mathrm{Arg}\,p-\pi$.

(37) For every p such that $\mathrm{Arg}\,p=0$ holds $p=[|p|,0]$ and $p_{\mathbf{2}}=0$.

(38) For every p such that $p\neq0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $\mathrm{Arg}\,p<\pi$ iff $\mathrm{Arg}(-p)\geq\pi$.

(39) For all p_1, p_2 such that $p_1 \neq p_2$ or $p_1 - p_2 \neq 0_{\mathcal{E}^2_{\rm T}}$ holds $\mathrm{Arg}(p_1 - p_2) < \pi$ iff $\mathrm{Arg}(p_2 - p_1) \geq \pi$.

(40) For every p holds $\mathrm{Arg}\, p \in]0,\pi[$ iff $p_{\mathbf{2}} > 0$.

(41) For every p such that $\mathrm{Arg}\, p \neq 0$ holds $\mathrm{Arg}\, p < \pi$ iff $\sin \mathrm{Arg}\, p > 0$.

(42) For all p_1, p_2 such that $\mathrm{Arg}\, p_1 < \pi$ and $\mathrm{Arg}\, p_2 < \pi$ holds $\mathrm{Arg}(p_1 + p_2) < \pi$.

Let us consider p_1, p_2, p_3. The functor $\measuredangle(p_1, p_2, p_3)$ yields a real number and is defined as follows:

(Def. 4) $\measuredangle(p_1, p_2, p_3) = \measuredangle(\mathrm{euc2cpx}(p_1), \mathrm{euc2cpx}(p_2), \mathrm{euc2cpx}(p_3))$.

The following propositions are true:

(43) For all p_1, p_2, p_3 holds $0 \leq \measuredangle(p_1, p_2, p_3)$ and $\measuredangle(p_1, p_2, p_3) < 2 \cdot \pi$.

(44) For all p_1, p_2, p_3 holds $\measuredangle(p_1, p_2, p_3) = \measuredangle(p_1 - p_2, 0_{\mathcal{E}^2_{\rm T}}, p_3 - p_2)$.

(45) For all p_1, p_2, p_3 such that $\measuredangle(p_1, p_2, p_3) = 0$ holds $\mathrm{Arg}(p_1 - p_2) = \mathrm{Arg}(p_3 - p_2)$ and $\measuredangle(p_3, p_2, p_1) = 0$.

(46) For all p_1, p_2, p_3 such that $\measuredangle(p_1, p_2, p_3) \neq 0$ holds $\measuredangle(p_3, p_2, p_1) = 2 \cdot \pi - \measuredangle(p_1, p_2, p_3)$.

(47) For all p_1, p_2, p_3 such that $\measuredangle(p_3, p_2, p_1) \neq 0$ holds $\measuredangle(p_3, p_2, p_1) = 2 \cdot \pi - \measuredangle(p_1, p_2, p_3)$.

(48) For all elements x, y of $\mathbb{C}$ holds $\Re((x|y)) = \Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)$.

(49) For all elements x, y of $\mathbb{C}$ holds $\Im((x|y)) = -\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y)$.

(50) For all p, q holds $|(p,q)| = p_{\mathbf{1}} \cdot q_{\mathbf{1}} + p_{\mathbf{2}} \cdot q_{\mathbf{2}}$.

(51) For all p_1, p_2 holds $|(p_1, p_2)| = \Re((\mathrm{euc2cpx}(p_1) | \mathrm{euc2cpx}(p_2)))$.

(52) For all p_1, p_2, p_3 such that $p_1 \neq 0_{\mathcal{E}^2_{\rm T}}$ and $p_2 \neq 0_{\mathcal{E}^2_{\rm T}}$ holds $|(p_1, p_2)| = 0$ iff $\measuredangle(p_1, 0_{\mathcal{E}^2_{\rm T}}, p_2) = \frac{\pi}{2}$ or $\measuredangle(p_1, 0_{\mathcal{E}^2_{\rm T}}, p_2) = \frac{3}{2} \cdot \pi$.

(53) Let given p_1, p_2. Suppose $p_1 \neq 0_{\mathcal{E}^2_{\rm T}}$ and $p_2 \neq 0_{\mathcal{E}^2_{\rm T}}$. Then $-(p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}} = |p_1| \cdot |p_2|$ or $-(p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}} = -|p_1| \cdot |p_2|$ if and only if $\measuredangle(p_1, 0_{\mathcal{E}^2_{\rm T}}, p_2) = \frac{\pi}{2}$ or $\measuredangle(p_1, 0_{\mathcal{E}^2_{\rm T}}, p_2) = \frac{3}{2} \cdot \pi$.

(54) For all p_1, p_2, p_3 such that $p_1 \neq p_2$ and $p_3 \neq p_2$ holds $|(p_1 - p_2, p_3 - p_2)| = 0$ iff $\measuredangle(p_1, p_2, p_3) = \frac{\pi}{2}$ or $\measuredangle(p_1, p_2, p_3) = \frac{3}{2} \cdot \pi$.

(55) For all p_1, p_2, p_3 such that $p_1 \neq p_2$ but $p_3 \neq p_2$ but $\measuredangle(p_1, p_2, p_3) = \frac{\pi}{2}$ or $\measuredangle(p_1, p_2, p_3) = \frac{3}{2} \cdot \pi$ holds $|p_1 - p_2|^{\mathbf{2}} + |p_3 - p_2|^{\mathbf{2}} = |p_1 - p_3|^{\mathbf{2}}$.

(56) For all p_1, p_2, p_3 such that $p_2 \neq p_1$ and $p_1 \neq p_3$ and $p_3 \neq p_2$ and $\measuredangle(p_2, p_1, p_3) < \pi$ and $\measuredangle(p_1, p_3, p_2) < \pi$ and $\measuredangle(p_3, p_2, p_1) < \pi$ holds $\measuredangle(p_2, p_1, p_3) + \measuredangle(p_1, p_3, p_2) + \measuredangle(p_3, p_2, p_1) = \pi$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor $\mathrm{Triangle}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 5) $\mathrm{Triangle}(p_1, p_2, p_3) = \mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, p_3) \cup \mathcal{L}(p_3, p_1)$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor $\mathrm{ClInsideOfTriangle}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 6) $\mathrm{ClInsideOfTriangle}(p_1, p_2, p_3) = \{p;\ p$ ranges over points of $\mathcal{E}^n_{\rm T}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} (0 \leq a_1 \wedge 0 \leq a_2 \wedge 0 \leq a_3 \wedge a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{InsideOfTriangle}(p_1, p_2, p_3)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 7) $\mathrm{InsideOfTriangle}(p_1, p_2, p_3) = \mathrm{ClInsideOfTriangle}(p_1, p_2, p_3) \setminus \mathrm{Triangle}(p_1, p_2, p_3)$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{OutsideOfTriangle}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by the condition (Def. 8).

(Def. 8) $\mathrm{OutsideOfTriangle}(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} ((0 > a_1 \vee 0 > a_2 \vee 0 > a_3) \wedge a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{plane}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 9) $\mathrm{plane}(p_1, p_2, p_3) = \mathrm{OutsideOfTriangle}(p_1, p_2, p_3) \cup \mathrm{ClInsideOfTriangle}(p_1, p_2, p_3)$.

Next we state two propositions:

(57) Let n be a natural number and p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p \in \mathrm{plane}(p_1, p_2, p_3)$. Then there exist real numbers a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = 1$ and $p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$.

(58) For every natural number n and for all points p_1, p_2, p_3 of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{Triangle}(p_1, p_2, p_3) \subseteq \mathrm{ClInsideOfTriangle}(p_1, p_2, p_3)$.

Let n be a natural number and let q_1, q_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. We say that q_1, q_2 are lindependent2 if and only if:

(Def. 10) For all real numbers a_1, a_2 such that $a_1 \cdot q_1 + a_2 \cdot q_2 = 0_{\mathcal{E}^n_{\mathrm{T}}}$ holds $a_1 = 0$ and $a_2 = 0$.

We introduce q_1, q_2 are ldependent2 as an antonym of q_1, q_2 are lindependent2.

We now state several propositions:

(59) Let n be a natural number and q_1, q_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If q_1, q_2 are lindependent2, then $q_1 \neq q_2$ and $q_1 \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ and $q_2 \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(60) Let n be a natural number and p_1, p_2, p_3, p_0 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p_0 \in \mathrm{plane}(p_1, p_2, p_3)$. Then there exist real numbers a_1, a_2, a_3 such that

(i) $p_0 = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$,

(ii) $a_1 + a_2 + a_3 = 1$, and

(iii) for all real numbers b_1, b_2, b_3 such that $p_0 = b_1 \cdot p_1 + b_2 \cdot p_2 + b_3 \cdot p_3$ and $b_1 + b_2 + b_3 = 1$ holds $b_1 = a_1$ and $b_2 = a_2$ and $b_3 = a_3$.

(61) Let n be a natural number and p_1, p_2, p_3, p_0 be points of $\mathcal{E}^n_{\mathrm{T}}$. Given real numbers a_1, a_2, a_3 such that $p_0 = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$ and $a_1 + a_2 + a_3 = 1$. Then $p_0 \in \mathrm{plane}(p_1, p_2, p_3)$.

(62) Let n be a natural number and p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. Then $\mathrm{plane}(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} (a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

(63) For all p_1, p_2, p_3 such that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 holds $\mathrm{plane}(p_1, p_2, p_3) = \mathcal{R}^2$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in \mathrm{plane}(p_1, p_2, p_3)$. The functor $\mathrm{tricord1}(p_1, p_2, p_3, p)$ yields a real number and is defined as follows:

(Def. 11) There exist real numbers a_2, a_3 such that $\mathrm{tricord1}(p_1, p_2, p_3, p) + a_2 + a_3 = 1$ and $p = \mathrm{tricord1}(p_1, p_2, p_3, p) \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in \mathrm{plane}(p_1, p_2, p_3)$. The functor tricord2$(p_1, p_2, p_3, p)$ yielding a real number is defined by:

(Def. 12) There exist real numbers a_1, a_3 such that $a_1 + \mathrm{tricord2}(p_1, p_2, p_3, p) + a_3 = 1$ and $p = a_1 \cdot p_1 + \mathrm{tricord2}(p_1, p_2, p_3, p) \cdot p_2 + a_3 \cdot p_3$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in \mathrm{plane}(p_1, p_2, p_3)$. The functor tricord2$(p_1, p_2, p_3, p)$ yields a real number and is defined by:

(Def. 13) There exist real numbers a_1, a_2 such that $a_1 + a_2 + \mathrm{tricord2}(p_1, p_2, p_3, p) = 1$ and $p = a_1 \cdot p_1 + a_2 \cdot p_2 + \mathrm{tricord2}(p_1, p_2, p_3, p) \cdot p_3$.

Let us consider p_1, p_2, p_3. The functor trcmap1(p_1, p_2, p_3) yielding a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 14) For every p holds $(\mathrm{trcmap1}(p_1, p_2, p_3))(p) = \mathrm{tricord1}(p_1, p_2, p_3, p)$.

Let us consider p_1, p_2, p_3. The functor trcmap2(p_1, p_2, p_3) yields a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined as follows:

(Def. 15) For every p holds $(\mathrm{trcmap2}(p_1, p_2, p_3))(p) = \mathrm{tricord2}(p_1, p_2, p_3, p)$.

Let us consider p_1, p_2, p_3. The functor trcmap3(p_1, p_2, p_3) yields a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 16) For every p holds $(\mathrm{trcmap3}(p_1, p_2, p_3))(p) = \mathrm{tricord2}(p_1, p_2, p_3, p)$.

Next we state several propositions:

(64) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in$ OutsideOfTriangle(p_1, p_2, p_3) if and only if one of the following conditions is satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) < 0$, or

(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) < 0$, or

(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) < 0$.

(65) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in$ Triangle(p_1, p_2, p_3) if and only if the following conditions are satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geq 0$,

(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) \geq 0$,

(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) \geq 0$, and

(iv) $\mathrm{tricord1}(p_1, p_2, p_3, p) = 0$ or $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$ or $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$.

(66) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in$ Triangle(p_1, p_2, p_3) if and only if one of the following conditions is satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) = 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geq 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geq 0$, or

(ii) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geq 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geq 0$, or

(iii) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geq 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geq 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$.

(67) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in$ InsideOfTriangle(p_1, p_2, p_3) if and only if the following conditions are satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) > 0$,

(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) > 0$, and

(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) > 0$.

(68) For all p_1, p_2, p_3 such that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 holds InsideOfTriangle(p_1, p_2, p_3) is non empty.

REFERENCES

[1] Kanchun and Yatsuka Nakamura. The inner product of finite sequences and of points of *n*-dimensional topological space. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol15/euclid_2.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Czesław Byliński. The complex numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/complex1.html.

[6] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.

[7] Wenpai Chang and Yatsuka Nakamura. Inner products and angles of complex numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol15/complex2.html.

[8] Library Committee. Introduction to arithmetic. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/arytm_0.html.

[9] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topmetr.html.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[15] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[16] Agnieszka Sakowicz, Jarosław Gryko, and Adam Grabowski. Sequences in $\mathcal{E}^N_{\mathrm{T}}$. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/toprns_1.html.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[18] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[20] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[21] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[22] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Journal of Formalized Mathematics*, 10, 1998. http://mizar.org/JFM/Vol10/sin_cos.html.

Received May 29, 2003

Published January 2, 2004