JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Inner Product of Finite Sequences and of Points of n-dimensional Topological Space

Kanchun
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

Summary. First, we define the inner product to finite sequences of real value. Next, we extend it to points of n-dimensional topological space $\mathcal{E}^n_{\mathrm{T}}$. At the end, orthogonality is introduced to this space.

MML Identifier: EUCLID_2.

WWW: http://mizar.org/JFM/Vol15/euclid_2.html

The articles [12], [3], [1], [10], [8], [2], [4], [7], [9], [5], [6], and [11] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we use the following convention: i, n are natural numbers, x, y, a are real numbers, v is an element of $\mathbb{R}^n$, and p, p_1, p_2, p_3, q, q_1, q_2 are points of $\mathcal{E}^n_{\mathrm{T}}$.

The following propositions are true:

(1) For every i such that $i \in \mathrm{Seg}\, n$ holds $(v \bullet \underbrace{\langle 0, \ldots, 0 \rangle}_{n})(i) = 0$.

(2) $v \bullet \underbrace{\langle 0, \ldots, 0 \rangle}_{n} = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$.

(3) For every finite sequence x of elements of $\mathbb{R}$ holds $(-1) \cdot x = -x$.

(4) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $x - y = x + -y$.

(5) For every finite sequence x of elements of $\mathbb{R}$ holds $\mathrm{len}(-x) = \mathrm{len}\, x$.

(6) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ holds $\mathrm{len}(x_1 + x_2) = \mathrm{len}\, x_1$.

(7) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ holds $\mathrm{len}(x_1 - x_2) = \mathrm{len}\, x_1$.

(8) For every real number a and for every finite sequence x of elements of $\mathbb{R}$ holds $\mathrm{len}(a \cdot x) = \mathrm{len}\, x$.

(9) For all finite sequences x, y, z of elements of $\mathbb{R}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ and $\mathrm{len}\, y = \mathrm{len}\, z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

2. Inner Product of Finite Sequences

Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$. The functor $|(x_1,x_2)|$ yields a real number and is defined by:

(Def. 1) $|(x_1,x_2)| = \sum(x_1 \bullet x_2)$.

Let us observe that the functor $|(x_1,x_2)|$ is commutative.

The following propositions are true:

(10) Let y_1, y_2 be finite sequences of elements of $\mathbb{R}$ and x_1, x_2 be elements of $\mathcal{R}^n$. If $x_1 = y_1$ and $x_2 = y_2$, then $|(y_1,y_2)| = \frac{1}{4} \cdot (|x_1+x_2|^{\mathbf{2}} - |x_1-x_2|^{\mathbf{2}})$.

(11) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x,x)| \geq 0$.

(12) For every finite sequence x of elements of $\mathbb{R}$ holds $|x|^{\mathbf{2}} = |(x,x)|$.

(13) For every finite sequence x of elements of $\mathbb{R}$ holds $|x| = \sqrt{|(x,x)|}$.

(14) For every finite sequence x of elements of $\mathbb{R}$ holds $0 \leq |x|$.

(15) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x,x)| = 0$ iff $x = \langle \underbrace{0,\ldots,0}_{\mathrm{len}\,x} \rangle$.

(16) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x,x)| = 0$ iff $|x| = 0$.

(17) For every finite sequence x of elements of $\mathbb{R}$ holds $|(x, \langle \underbrace{0,\ldots,0}_{\mathrm{len}\,x} \rangle)| = 0$.

(18) For every finite sequence x of elements of $\mathbb{R}$ holds $|(\langle \underbrace{0,\ldots,0}_{\mathrm{len}\,x} \rangle, x)| = 0$.

(19) For all finite sequences x, y, z of elements of $\mathbb{R}$ such that $\mathrm{len}\,x = \mathrm{len}\,y$ and $\mathrm{len}\,y = \mathrm{len}\,z$ holds $|(x+y,z)| = |(x,z)| + |(y,z)|$.

(20) For all finite sequences x, y of elements of $\mathbb{R}$ and for every real number a such that $\mathrm{len}\,x = \mathrm{len}\,y$ holds $|(a \cdot x,y)| = a \cdot |(x,y)|$.

(21) For all finite sequences x, y of elements of $\mathbb{R}$ and for every real number a such that $\mathrm{len}\,x = \mathrm{len}\,y$ holds $|(x,a \cdot y)| = a \cdot |(x,y)|$.

(22) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\,x_1 = \mathrm{len}\,x_2$ holds $|(-x_1,x_2)| = -|(x_1,x_2)|$.

(23) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\,x_1 = \mathrm{len}\,x_2$ holds $|(x_1,-x_2)| = -|(x_1,x_2)|$.

(24) For all finite sequences x_1, x_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\,x_1 = \mathrm{len}\,x_2$ holds $|(-x_1,-x_2)| = |(x_1,x_2)|$.

(25) For all finite sequences x_1, x_2, x_3 of elements of $\mathbb{R}$ such that $\mathrm{len}\,x_1 = \mathrm{len}\,x_2$ and $\mathrm{len}\,x_2 = \mathrm{len}\,x_3$ holds $|(x_1-x_2,x_3)| = |(x_1,x_3)| - |(x_2,x_3)|$.

(26) Let x, y be real numbers and x_1, x_2, x_3 be finite sequences of elements of $\mathbb{R}$. If $\mathrm{len}\,x_1 = \mathrm{len}\,x_2$ and $\mathrm{len}\,x_2 = \mathrm{len}\,x_3$, then $|(x \cdot x_1 + y \cdot x_2,x_3)| = x \cdot |(x_1,x_3)| + y \cdot |(x_2,x_3)|$.

(27) For all finite sequences x, y_1, y_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\,x = \mathrm{len}\,y_1$ and $\mathrm{len}\,y_1 = \mathrm{len}\,y_2$ holds $|(x,y_1+y_2)| = |(x,y_1)| + |(x,y_2)|$.

(28) For all finite sequences x, y_1, y_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\,x = \mathrm{len}\,y_1$ and $\mathrm{len}\,y_1 = \mathrm{len}\,y_2$ holds $|(x,y_1-y_2)| = |(x,y_1)| - |(x,y_2)|$.

(29) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{R}$. If $\mathrm{len}\,x_1 = \mathrm{len}\,x_2$ and $\mathrm{len}\,x_2 = \mathrm{len}\,y_1$ and $\mathrm{len}\,y_1 = \mathrm{len}\,y_2$, then $|(x_1+x_2,y_1+y_2)| = |(x_1,y_1)| + |(x_1,y_2)| + |(x_2,y_1)| + |(x_2,y_2)|$.

(30) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{R}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1 - x_2, y_1 - y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(31) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x+y, x+y)| = |(x,x)| + 2 \cdot |(x,y)| + |(y,y)|$.

(32) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x-y, x-y)| = (|(x,x)| - 2 \cdot |(x,y)|) + |(y,y)|$.

(33) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x+y|^{\mathbf{2}} = |x|^{\mathbf{2}} + 2 \cdot |(y,x)| + |y|^{\mathbf{2}}$.

(34) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x-y|^{\mathbf{2}} = (|x|^{\mathbf{2}} - 2 \cdot |(y,x)|) + |y|^{\mathbf{2}}$.

(35) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x+y|^{\mathbf{2}} + |x-y|^{\mathbf{2}} = 2 \cdot (|x|^{\mathbf{2}} + |y|^{\mathbf{2}})$.

(36) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x+y|^{\mathbf{2}} - |x-y|^{\mathbf{2}} = 4 \cdot |(x,y)|$.

(37) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $||(x,y)|| \leq |x| \cdot |y|$.

(38) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|x+y| \leq |x| + |y|$.

3. Inner Product of Points of $\mathcal{E}^n_{\mathrm{T}}$

Let us consider n and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|(p,q)|$ yields a real number and is defined as follows:

(Def. 2) There exist finite sequences f, g of elements of $\mathbb{R}$ such that $f = p$ and $g = q$ and $|(p,q)| = |(f,g)|$.

Let us observe that the functor $|(p,q)|$ is commutative.

Next we state a number of propositions:

(39) For every natural number n and for all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ holds $|(p_1, p_2)| = \frac{1}{4} \cdot (|p_1 + p_2|^{\mathbf{2}} - |p_1 - p_2|^{\mathbf{2}})$.

(40) $|(p_1 + p_2, p_3)| = |(p_1, p_3)| + |(p_2, p_3)|$.

(41) For every real number x holds $|(x \cdot p_1, p_2)| = x \cdot |(p_1, p_2)|$.

(42) For every real number x holds $|(p_1, x \cdot p_2)| = x \cdot |(p_1, p_2)|$.

(43) $|(-p_1, p_2)| = -|(p_1, p_2)|$.

(44) $|(p_1, -p_2)| = -|(p_1, p_2)|$.

(45) $|(-p_1, -p_2)| = |(p_1, p_2)|$.

(46) $|(p_1 - p_2, p_3)| = |(p_1, p_3)| - |(p_2, p_3)|$.

(47) $|(x \cdot p_1 + y \cdot p_2, p_3)| = x \cdot |(p_1, p_3)| + y \cdot |(p_2, p_3)|$.

(48) $|(p, q_1 + q_2)| = |(p, q_1)| + |(p, q_2)|$.

(49) $|(p, q_1 - q_2)| = |(p, q_1)| - |(p, q_2)|$.

(50) $|(p_1 + p_2, q_1 + q_2)| = |(p_1, q_1)| + |(p_1, q_2)| + |(p_2, q_1)| + |(p_2, q_2)|$.

(51) $|(p_1 - p_2, q_1 - q_2)| = (|(p_1, q_1)| - |(p_1, q_2)| - |(p_2, q_1)|) + |(p_2, q_2)|$.

(52) $|(p+q, p+q)| = |(p,p)| + 2 \cdot |(p,q)| + |(q,q)|$.

(53) $|(p-q,p-q)| = (|(p,p)| - 2\cdot|(p,q)|) + |(q,q)|$.

(54) $|(p,0_{\mathcal{E}^n_{\rm T}})| = 0$.

(55) $|(0_{\mathcal{E}^n_{\rm T}},p)| = 0$.

(56) $|(0_{\mathcal{E}^n_{\rm T}},0_{\mathcal{E}^n_{\rm T}})| = 0$.

(57) $|(p,p)| \geq 0$.

(58) $|(p,p)| = |p|^{\mathbf{2}}$.

(59) $|p| = \sqrt{|(p,p)|}$.

(60) $0 \leq |p|$.

(61) $|0_{\mathcal{E}^n_{\rm T}}| = 0$.

(62) $|(p,p)| = 0$ iff $|p| = 0$.

(63) $|(p,p)| = 0$ iff $p = 0_{\mathcal{E}^n_{\rm T}}$.

(64) $|p| = 0$ iff $p = 0_{\mathcal{E}^n_{\rm T}}$.

(65) $p \neq 0_{\mathcal{E}^n_{\rm T}}$ iff $|(p,p)| > 0$.

(66) $p \neq 0_{\mathcal{E}^n_{\rm T}}$ iff $|p| > 0$.

(67) $|p+q|^{\mathbf{2}} = |p|^{\mathbf{2}} + 2\cdot|(q,p)| + |q|^{\mathbf{2}}$.

(68) $|p-q|^{\mathbf{2}} = (|p|^{\mathbf{2}} - 2\cdot|(q,p)|) + |q|^{\mathbf{2}}$.

(69) $|p+q|^{\mathbf{2}} + |p-q|^{\mathbf{2}} = 2\cdot(|p|^{\mathbf{2}} + |q|^{\mathbf{2}})$.

(70) $|p+q|^{\mathbf{2}} - |p-q|^{\mathbf{2}} = 4\cdot|(p,q)|$.

(71) $|(p,q)| = \frac{1}{4}\cdot(|p+q|^{\mathbf{2}} - |p-q|^{\mathbf{2}})$.

(72) $|(p,q)| \leq |(p,p)| + |(q,q)|$.

(73) For all points p, q of $\mathcal{E}^n_{\rm T}$ holds $||(p,q)|| \leq |p|\cdot|q|$.

(74) $|p+q| \leq |p| + |q|$.

Let us consider n, p, q. We say that p, q are orthogonal if and only if:

(Def. 3) $|(p,q)| = 0$.

Let us note that the predicate p, q are orthogonal is symmetric.

Next we state several propositions:

(75) p, $0_{\mathcal{E}^n_{\rm T}}$ are orthogonal.

(76) $0_{\mathcal{E}^n_{\rm T}}$, p are orthogonal.

(77) p, p are orthogonal iff $p = 0_{\mathcal{E}^n_{\rm T}}$.

(78) If p, q are orthogonal, then $a\cdot p$, q are orthogonal.

(79) If p, q are orthogonal, then p, $a\cdot q$ are orthogonal.

(80) If for every q holds p, q are orthogonal, then $p = 0_{\mathcal{E}^n_{\rm T}}$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[5] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.

[6] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[8] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[9] Agnieszka Sakowicz, Jarosław Gryko, and Adam Grabowski. Sequences in $\mathcal{E}^N_{\mathrm{T}}$. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/toprns_1.html.

[10] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[11] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

Received February 3, 2003

Published January 2, 2004