JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

Equations in Many Sorted Algebras

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Summary. This paper is preparation to prove Birkhoff's Theorem. Some properties of many sorted algebras are proved. The last section of this work shows that every equation valid in a many sorted algebra is also valid in each subalgebra, and each image of it. Moreover for a family of many sorted algebras $(A_i : i \in I)$ if every equation is valid in each A_i, $i \in I$ then is also valid in product $\prod(A_i : i \in I)$.

MML Identifier: `EQUATION`.

WWW: `http://mizar.org/JFM/Vol9/equation.html`

The articles [20], [8], [25], [24], [26], [5], [7], [6], [21], [10], [3], [9], [1], [22], [23], [15], [16], [17], [4], [13], [14], [12], [19], [18], [11], and [2] provide the notation and terminology for this paper.

1. On the Functions and Many Sorted Functions

In this paper I denotes a set.

The following propositions are true:

(1) Let A be a set, B, C be non empty sets, f be a function from A into B, and g be a function from B into C. If $\operatorname{rng}(g \cdot f) = C$, then $\operatorname{rng} g = C$.

(2) Let A be a many sorted set indexed by I, B, C be non-empty many sorted sets indexed by I, f be a many sorted function from A into B, and g be a many sorted function from B into C. If $g \circ f$ is onto, then g is onto.

(3) Let A, B be non empty sets, C, y be sets, and f be a function. If $f \in (C^B)^A$ and $y \in B$, then $\operatorname{dom}(\operatorname{commute}(f))(y) = A$ and $\operatorname{rng}(\operatorname{commute}(f))(y) \subseteq C$.

(5)[1] Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let f be a many sorted function indexed by I. If $\operatorname{dom}_\kappa f(\kappa) = A$ and $\operatorname{rng}_\kappa f(\kappa) \subseteq B$, then f is a many sorted function from A into B.

(6) Let A, B be many sorted sets indexed by I, F be a many sorted function from A into B, C, E be many sorted subsets indexed by A, and D be a many sorted subset indexed by C. If $E = D$, then $F \restriction C \restriction D = F \restriction E$.

(7) Let B be a non-empty many sorted set indexed by I, C be a many sorted set indexed by I, A be a many sorted subset indexed by C, and F be a many sorted function from A into B. Then there exists a many sorted function G from C into B such that $G \restriction A = F$.

[1] The proposition (4) has been removed.

Let I be a set, let A be a many sorted set indexed by I, and let F be a many sorted function indexed by I. The functor $F^{-1}(A)$ yields a many sorted set indexed by I and is defined as follows:

(Def. 1) For every set i such that $i \in I$ holds $(F^{-1}(A))(i) = F(i)^{-1}(A(i))$.

Next we state a number of propositions:

(8) Let A, B, C be many sorted sets indexed by I and F be a many sorted function from A into B. Then $F^{\circ}C$ is a many sorted subset indexed by B.

(9) Let A, B, C be many sorted sets indexed by I and F be a many sorted function from A into B. Then $F^{-1}(C)$ is a many sorted subset indexed by A.

(10) Let A, B be many sorted sets indexed by I and F be a many sorted function from A into B. If F is onto, then $F^{\circ}A = B$.

(11) Let A, B be many sorted sets indexed by I and F be a many sorted function from A into B. If A is transformable to B, then $F^{-1}(B) = A$.

(12) Let A be a many sorted set indexed by I and F be a many sorted function indexed by I. If $A \subseteq \mathrm{rng}_{\kappa} F(\kappa)$, then $F^{\circ} F^{-1}(A) = A$.

(13) For every many sorted function f indexed by I and for every many sorted set X indexed by I holds $f^{\circ}X \subseteq \mathrm{rng}_{\kappa} f(\kappa)$.

(14) For every many sorted function f indexed by I holds $f^{\circ}(\mathrm{dom}_{\kappa} f(\kappa)) = \mathrm{rng}_{\kappa} f(\kappa)$.

(15) For every many sorted function f indexed by I holds $f^{-1}(\mathrm{rng}_{\kappa} f(\kappa)) = \mathrm{dom}_{\kappa} f(\kappa)$.

(16) For every many sorted set A indexed by I holds $(\mathrm{id}_A)^{\circ}A = A$.

(17) For every many sorted set A indexed by I holds $(\mathrm{id}_A)^{-1}(A) = A$.

2. On the Many Sorted Algebras

In the sequel S is a non empty non void many sorted signature and U_0, U_1 are non-empty algebras over S.

One can prove the following propositions:

(19)[2] Every algebra A over S is a subalgebra of the algebra of A.

(20) Let U_0 be an algebra over S, A be a subalgebra of U_0, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o,A)$, then $x \in \mathrm{Args}(o,U_0)$.

(21) Let U_0 be an algebra over S, A be a subalgebra of U_0, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o,A)$, then $(\mathrm{Den}(o,A))(x) = (\mathrm{Den}(o,U_0))(x)$.

(22) Let F be an algebra family of I over S, B be a subalgebra of $\prod F$, o be an operation symbol of S, and x be a set. If $x \in \mathrm{Args}(o,B)$, then $(\mathrm{Den}(o,B))(x)$ is a function and $(\mathrm{Den}(o,\prod F))(x)$ is a function.

Let S be a non void non empty many sorted signature, let A be an algebra over S, and let B be a subalgebra of A. The functor SuperAlgebraSet(B) is defined by the condition (Def. 2).

(Def. 2) Let x be a set. Then $x \in \mathrm{SuperAlgebraSet}(B)$ if and only if there exists a strict subalgebra C of A such that $x = C$ and B is a subalgebra of C.

[2] The proposition (18) has been removed.

Let S be a non void non empty many sorted signature, let A be an algebra over S, and let B be a subalgebra of A. Observe that SuperAlgebraSet(B) is non empty.

Let S be a non empty non void many sorted signature. Note that there exists an algebra over S which is strict, non-empty, and free.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let X be a non-empty locally-finite subset of A. One can check that $\mathrm{Gen}(X)$ is finitely-generated.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Note that there exists a subalgebra of A which is strict, non-empty, and finitely-generated.

Let S be a non empty non void many sorted signature and let A be a feasible algebra over S. Observe that there exists a subalgebra of A which is feasible.

We now state several propositions:

(23) Let A be an algebra over S, C be a subalgebra of A, and D be a many sorted subset indexed by the sorts of A. Suppose $D =$ the sorts of C. Let h be a many sorted function from A into U_0 and g be a many sorted function from C into U_0. Suppose $g = h \restriction D$. Let o be an operation symbol of S, x be an element of $\mathrm{Args}(o,A)$, and y be an element of $\mathrm{Args}(o,C)$. If $\mathrm{Args}(o,C) \neq \emptyset$ and $x = y$, then $h\#x = g\#y$.

(24) Let A be a feasible algebra over S, C be a feasible subalgebra of A, and D be a many sorted subset indexed by the sorts of A. Suppose $D =$ the sorts of C. Let h be a many sorted function from A into U_0. Suppose h is a homomorphism of A into U_0. Let g be a many sorted function from C into U_0. If $g = h \restriction D$, then g is a homomorphism of C into U_0.

(25) Let B be a strict non-empty algebra over S, G be a generator set of U_0, H be a non-empty generator set of B, and f be a many sorted function from U_0 into B. Suppose $H \subseteq f^\circ G$ and f is a homomorphism of U_0 into B. Then f is an epimorphism of U_0 onto B.

(26) Let W be a strict free non-empty algebra over S and F be a many sorted function from U_0 into U_1. Suppose F is an epimorphism of U_0 onto U_1. Let G be a many sorted function from W into U_1. Suppose G is a homomorphism of W into U_1. Then there exists a many sorted function H from W into U_0 such that H is a homomorphism of W into U_0 and $G = F \circ H$.

(27) Let I be a non empty finite set, A be a non-empty algebra over S, and F be an algebra family of I over S. Suppose that for every element i of I there exists a strict non-empty finitely-generated subalgebra C of A such that $C = F(i)$. Then there exists a strict non-empty finitely-generated subalgebra B of A such that for every element i of I holds $F(i)$ is a subalgebra of B.

(28) Let A, B be strict non-empty finitely-generated subalgebras of U_0. Then there exists a strict non-empty finitely-generated subalgebra M of U_0 such that A is a subalgebra of M and B is a subalgebra of M.

(29) Let S_1 be a non empty non void many sorted signature, A_1 be a non-empty algebra over S_1, and C be a set. Suppose $C = \{A; A$ ranges over elements of $\mathrm{Subalgebras}(A_1)$: $\bigvee_{R:\ \text{strict non-empty finitely-generated subalgebra of } A_1} R = A\}$. Let F be an algebra family of C over S_1. Suppose that for every set c such that $c \in C$ holds $c = F(c)$. Then there exists a strict non-empty subalgebra P_1 of $\prod F$ such that there exists a many sorted function from P_1 into A_1 which is an epimorphism of P_1 onto A_1.

(30) Let U_0 be a feasible free algebra over S, A be a free generator set of U_0, and Z be a subset of U_0. If $Z \subseteq A$ and $\mathrm{Gen}(Z)$ is feasible, then $\mathrm{Gen}(Z)$ is free.

3. Equations in Many Sorted Algebras

Let S be a non empty non void many sorted signature. The functor $\mathrm{T}_S(\mathbb{N})$ yields an algebra over S and is defined as follows:

(Def. 3) $\mathrm{T}_S(\mathbb{N}) = \mathrm{Free}((\text{the carrier of } S) \longmapsto \mathbb{N})$.

Let S be a non empty non void many sorted signature. One can verify that $\mathrm{T}_S(\mathbb{N})$ is strict, non-empty, and free.

Let S be a non empty non void many sorted signature. The equations of S constitute a many sorted set indexed by the carrier of S defined by:

(Def. 4) The equations of $S = [\![$the sorts of $\mathrm{T}_S(\mathbb{N})$, the sorts of $\mathrm{T}_S(\mathbb{N})]\!]$.

Let S be a non empty non void many sorted signature. Note that the equations of S is non-empty.

Let S be a non empty non void many sorted signature. A set of equations of S is a many sorted subset indexed by the equations of S.

In the sequel s denotes a sort symbol of S, e denotes an element of (the equations of $S)(s)$, and E denotes a set of equations of S.

Let S be a non empty non void many sorted signature, let s be a sort symbol of S, and let x, y be elements of (the sorts of $\mathrm{T}_S(\mathbb{N}))(s)$. Then $\langle x, y\rangle$ is an element of (the equations of $S)(s)$. We introduce x=y as a synonym of $\langle x, y\rangle$.

Next we state two propositions:

(31) $e_{\mathbf{1}} \in$ (the sorts of $\mathrm{T}_S(\mathbb{N}))(s)$.

(32) $e_{\mathbf{2}} \in$ (the sorts of $\mathrm{T}_S(\mathbb{N}))(s)$.

Let S be a non empty non void many sorted signature, let A be an algebra over S, let s be a sort symbol of S, and let e be an element of (the equations of $S)(s)$. The predicate $A \models e$ is defined by:

(Def. 5) For every many sorted function h from $\mathrm{T}_S(\mathbb{N})$ into A such that h is a homomorphism of $\mathrm{T}_S(\mathbb{N})$ into A holds $h(s)(e_{\mathbf{1}}) = h(s)(e_{\mathbf{2}})$.

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let E be a set of equations of S. The predicate $A \models E$ is defined as follows:

(Def. 6) For every sort symbol s of S and for every element e of (the equations of $S)(s)$ such that $e \in E(s)$ holds $A \models e$.

The following propositions are true:

(33) For every strict non-empty subalgebra U_2 of U_0 such that $U_0 \models e$ holds $U_2 \models e$.

(34) For every strict non-empty subalgebra U_2 of U_0 such that $U_0 \models E$ holds $U_2 \models E$.

(35) If U_0 and U_1 are isomorphic and $U_0 \models e$, then $U_1 \models e$.

(36) If U_0 and U_1 are isomorphic and $U_0 \models E$, then $U_1 \models E$.

(37) For every congruence R of U_0 such that $U_0 \models e$ holds $U_0/R \models e$.

(38) For every congruence R of U_0 such that $U_0 \models E$ holds $U_0/R \models E$.

(39) Let F be an algebra family of I over S. Suppose that for every set i such that $i \in I$ there exists an algebra A over S such that $A = F(i)$ and $A \models e$. Then $\prod F \models e$.

(40) Let F be an algebra family of I over S. Suppose that for every set i such that $i \in I$ there exists an algebra A over S such that $A = F(i)$ and $A \models E$. Then $\prod F \models E$.

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Received May 30, 1997

Published January 2, 2004