JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

On the Monoid of Endomorphisms of Universal Algebra and Many Sorted Algebra

Jarosław Gryko
Warsaw University
Białystok

MML Identifier: ENDALG.

WWW: http://mizar.org/JFM/Vol7/endalg.html

The articles [13], [7], [16], [17], [4], [1], [12], [3], [6], [5], [14], [2], [11], [15], [9], [10], and [8] provide the notation and terminology for this paper.

In this paper U_1 is a universal algebra.

Let us consider U_1. The functor $\mathrm{end}(U_1)$ yielding a non empty set of functions from the carrier of U_1 to the carrier of U_1 is defined by:

(Def. 1) For every function h from U_1 into U_1 holds $h \in \mathrm{end}(U_1)$ iff h is a homomorphism of U_1 into U_1.

Next we state three propositions:

(1) $\mathrm{end}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

(3)[1] $\mathrm{id}_{\text{the carrier of } U_1} \in \mathrm{end}(U_1)$.

(4) For all elements f_1, f_2 of $\mathrm{end}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{end}(U_1)$.

Let us consider U_1. The functor $\mathrm{Comp}(U_1)$ yields a binary operation on $\mathrm{end}(U_1)$ and is defined by:

(Def. 2) For all elements x, y of $\mathrm{end}(U_1)$ holds $(\mathrm{Comp}(U_1))(x, y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{End}(U_1)$ yields a strict multiplicative loop structure and is defined by:

(Def. 3) The carrier of $\mathrm{End}(U_1) = \mathrm{end}(U_1)$ and the multiplication of $\mathrm{End}(U_1) = \mathrm{Comp}(U_1)$ and the unity of $\mathrm{End}(U_1) = \mathrm{id}_{\text{the carrier of } U_1}$.

Let us consider U_1. One can verify that $\mathrm{End}(U_1)$ is non empty.

Let us consider U_1. Observe that $\mathrm{End}(U_1)$ is left unital, well unital, and associative.

We now state two propositions:

(5) For all elements x, y of $\mathrm{End}(U_1)$ and for all elements f, g of $\mathrm{end}(U_1)$ such that $x = f$ and $y = g$ holds $x \cdot y = g \cdot f$.

[1] The proposition (2) has been removed.

(6) $\mathrm{id}_{\text{the carrier of } U_1} = \mathbf{1}_{\mathrm{End}(U_1)}$.

In the sequel S denotes a non void non empty many sorted signature and U_2 denotes a non-empty algebra over S.

Let us consider S, U_2. The functor $\mathrm{end}(U_2)$ yielding a set of many sorted functions from the sorts of U_2 into the sorts of U_2 is defined by the conditions (Def. 4).

(Def. 4)(i) Every element of $\mathrm{end}(U_2)$ is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in \mathrm{end}(U_2)$ iff h is a homomorphism of U_2 into U_2.

The following four propositions are true:

(9)[2] $\mathrm{end}(U_2) \subseteq \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(10) $\mathrm{id}_{\text{the sorts of } U_2} \in \mathrm{end}(U_2)$.

(11) For all elements f_1, f_2 of $\mathrm{end}(U_2)$ holds $f_1 \circ f_2 \in \mathrm{end}(U_2)$.

(12) For every many sorted function F from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$ and for every element f of $\mathrm{end}(U_1)$ such that $F = \{0\} \longmapsto f$ holds $F \in \mathrm{end}(\mathrm{MSAlg}(U_1))$.

Let us consider S, U_2. The functor $\mathrm{Comp}(U_2)$ yielding a binary operation on $\mathrm{end}(U_2)$ is defined by:

(Def. 5) For all elements x, y of $\mathrm{end}(U_2)$ holds $(\mathrm{Comp}(U_2))(x, y) = y \circ x$.

Let us consider S, U_2. The functor $\mathrm{End}(U_2)$ yields a strict multiplicative loop structure and is defined by:

(Def. 6) The carrier of $\mathrm{End}(U_2) = \mathrm{end}(U_2)$ and the multiplication of $\mathrm{End}(U_2) = \mathrm{Comp}(U_2)$ and the unity of $\mathrm{End}(U_2) = \mathrm{id}_{\text{the sorts of } U_2}$.

Let us consider S, U_2. Observe that $\mathrm{End}(U_2)$ is non empty.

Let us consider S, U_2. One can check that $\mathrm{End}(U_2)$ is left unital, well unital, and associative.

We now state three propositions:

(13) For all elements x, y of $\mathrm{End}(U_2)$ and for all elements f, g of $\mathrm{end}(U_2)$ such that $x = f$ and $y = g$ holds $x \cdot y = g \circ f$.

(14) $\mathrm{id}_{\text{the sorts of } U_2} = \mathbf{1}_{\mathrm{End}(U_2)}$.

(16)[3] For every element f of $\mathrm{end}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

Let G, H be non empty groupoids and let I_1 be a map from G into H. We say that I_1 is multiplicative if and only if:

(Def. 7) For all elements x, y of G holds $I_1(x \cdot y) = I_1(x) \cdot I_1(y)$.

Let G, H be non empty multiplicative loop structures and let I_1 be a map from G into H. We say that I_1 is unity-preserving if and only if:

(Def. 8) $I_1(\mathbf{1}_G) = \mathbf{1}_H$.

Let us note that there exists a non empty multiplicative loop structure which is left unital.

Let G, H be left unital non empty multiplicative loop structures. Observe that there exists a map from G into H which is multiplicative and unity-preserving.

Let G, H be left unital non empty multiplicative loop structures. A homomorphism from G to H is a multiplicative unity-preserving map from G into H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is a monomorphism if and only if:

[2] The propositions (7) and (8) have been removed.

[3] The proposition (15) has been removed.

(Def. 9) h is one-to-one.

We say that h is an epimorphism if and only if:

(Def. 10) $\operatorname{rng} h =$ the carrier of H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is an isomorphism if and only if:

(Def. 11) h is an epimorphism and a monomorphism.

The following proposition is true

(17) Let G be a left unital non empty multiplicative loop structure. Then $\mathrm{id}_{\text{the carrier of } G}$ is a homomorphism from G to G.

Let G, H be left unital non empty multiplicative loop structures. We say that G and H are isomorphic if and only if:

(Def. 12) There exists a homomorphism from G to H which is an isomorphism.

Let us note that the predicate G and H are isomorphic is reflexive.

Next we state three propositions:

(18) Let h be a function. Suppose $\operatorname{dom} h = \operatorname{end}(U_1)$ and for every set x such that $x \in \operatorname{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\operatorname{End}(U_1)$ to $\operatorname{End}(\operatorname{MSAlg}(U_1))$.

(19) Let h be a homomorphism from $\operatorname{End}(U_1)$ to $\operatorname{End}(\operatorname{MSAlg}(U_1))$. If for every set x such that $x \in \operatorname{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$, then h is an isomorphism.

(20) $\operatorname{End}(U_1)$ and $\operatorname{End}(\operatorname{MSAlg}(U_1))$ are isomorphic.

REFERENCES

[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[8] Artur Korniłowicz. On the group of automorphisms of universal algebra and many sorted algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/autalg_1.html.

[9] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/alg_1.html.

[10] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[11] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/unialg_1.html.

[12] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[14] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[15] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[17] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received October 17, 1995

Published January 2, 2004