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Ordered Affine Spaces Defined in Terms of Directed Parallelity — Part I[1]

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Summary. In the article we consider several geometrical relations in given arbitrary ordered affine space defined in terms of directed parallelity. In particular we introduce the notions of the nondirected parallelity of segments, of collinearity, and the betweenness relation determined by the given relation of directed parallelity. The obtained structures satisfy commonly accepted axioms for affine spaces. At the end of the article we introduce a formal definition of affine space and affine plane (defined in terms of parallelity of segments).

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The articles [5], [2], [6], [4], [7], [3], and [1] provide the notation and terminology for this paper.

In this paper X denotes a non empty set.

Let us consider X and let R be a binary relation on $[: X, X :]$. The functor $\lambda(R)$ yields a binary relation on $[: X, X :]$ and is defined as follows:

(Def. 1) For all elements a, b, c, d of X holds $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in \lambda(R)$ iff $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R$ or $\langle\langle a, b\rangle, \langle d, c\rangle\rangle \in R$.

Let S be a non empty affine structure. The functor $\Lambda(S)$ yielding a strict affine structure is defined as follows:

(Def. 2) $\Lambda(S) = \langle$the carrier of S, $\lambda($the congruence of $S)\rangle$.

Let S be a non empty affine structure. Observe that $\Lambda(S)$ is non empty.

We follow the rules: S denotes an ordered affine space and a, b, c, d, x, y, z, t, u, w denote elements of S.

Next we state several propositions:

(4)[1] $x,y \upuparrows x,y$.

(5) If $x,y \upuparrows z,t$, then $y,x \upuparrows t,z$ and $z,t \upuparrows x,y$ and $t,z \upuparrows y,x$.

(6) If $z \neq t$ and $x,y \upuparrows z,t$ and $z,t \upuparrows u,w$, then $x,y \upuparrows u,w$.

(7) $x,x \upuparrows y,z$ and $y,z \upuparrows x,x$.

(8) If $x,y \upuparrows z,t$ and $x,y \upuparrows t,z$, then $x = y$ or $z = t$.

(9) $x,y \upuparrows x,z$ iff $x,y \upuparrows y,z$ or $x,z \upuparrows z,y$.

[1]Supported by RPBP.III-24.C2.
[1] The propositions (1)–(3) have been removed.

Let S be a non empty affine structure and let a, b, c be elements of S. We say that b is a midpoint of a, c if and only if:

(Def. 3) $a,b \upharpoonleft\!\downharpoonright b,c$.

We now state a number of propositions:

(11)[2] $x,y \upharpoonleft\!\downharpoonright x,z$ iff y is a midpoint of x, z or z is a midpoint of x, y.

(12) If b is a midpoint of a, a, then $a = b$.

(13) If b is a midpoint of a, c, then b is a midpoint of c, a.

(14) x is a midpoint of x, y and y is a midpoint of x, y.

(15) If b is a midpoint of a, c and c is a midpoint of a, d, then c is a midpoint of b, d.

(16) If $b \neq c$ and b is a midpoint of a, c and c is a midpoint of b, d, then c is a midpoint of a, d.

(17) There exists z such that y is a midpoint of x, z and $y \neq z$.

(18) If y is a midpoint of x, z and x is a midpoint of y, z, then $x = y$.

(19) Suppose $x \neq y$ and y is a midpoint of x, z and y is a midpoint of x, t. Then z is a midpoint of y, t or t is a midpoint of y, z.

(20) Suppose $x \neq y$ and y is a midpoint of x, z and y is a midpoint of x, t. Then z is a midpoint of x, t or t is a midpoint of x, z.

(21) Suppose y is a midpoint of x, t and z is a midpoint of x, t. Then y is a midpoint of x, z or z is a midpoint of x, y.

Let S be a non empty affine structure and let a, b, c, d be elements of S. The predicate $a,b \parallel c,d$ is defined as follows:

(Def. 4) $a,b \upharpoonleft\!\downharpoonright c,d$ or $a,b \upharpoonleft\!\downharpoonright d,c$.

The following propositions are true:

(23)[3] $a,b \parallel c,d$ iff $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in \lambda(\text{the congruence of } S)$.

(24) $x,y \parallel y,x$ and $x,y \parallel x,y$.

(25) $x,y \parallel z,z$ and $z,z \parallel x,y$.

(26) If $x,y \parallel x,z$, then $y,x \parallel y,z$.

(27) If $x,y \parallel z,t$, then $x,y \parallel t,z$ and $y,x \parallel z,t$ and $y,x \parallel t,z$ and $z,t \parallel x,y$ and $z,t \parallel y,x$ and $t,z \parallel x,y$ and $t,z \parallel y,x$.

(28) If $a \neq b$ and if $a,b \parallel x,y$ and $a,b \parallel z,t$ or $a,b \parallel x,y$ and $z,t \parallel a,b$ or $x,y \parallel a,b$ and $z,t \parallel a,b$ or $x,y \parallel a,b$ and $a,b \parallel z,t$, then $x,y \parallel z,t$.

(29) There exist x, y, z such that $x,y \nparallel x,z$.

(30) There exists t such that $x,z \parallel y,t$ and $y \neq t$.

(31) There exists t such that $x,y \parallel z,t$ and $x,z \parallel y,t$.

(32) If $z,x \parallel x,t$ and $x \neq z$, then there exists u such that $y,x \parallel x,u$ and $y,z \parallel t,u$.

Let S be a non empty affine structure and let a, b, c be elements of S. The predicate $\mathbf{L}(a,b,c)$ is defined as follows:

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[2] The proposition (10) has been removed.
[3] The proposition (22) has been removed.

(Def. 5) $a, b \parallel a, c$.

We introduce *a*, *b* and *c* are collinear as a synonym of $\mathbf{L}(a, b, c)$.

We now state a number of propositions:

(34)[4] If *b* is a midpoint of *a*, *c*, then *a*, *b* and *c* are collinear.

(35) Suppose *a*, *b* and *c* are collinear. Then *b* is a midpoint of *a*, *c* or *a* is a midpoint of *b*, *c* or *c* is a midpoint of *a*, *b*.

(36) Suppose *x*, *y* and *z* are collinear. Then

(i) *x*, *z* and *y* are collinear,

(ii) *y*, *x* and *z* are collinear,

(iii) *y*, *z* and *x* are collinear,

(iv) *z*, *x* and *y* are collinear, and

(v) *z*, *y* and *x* are collinear.

(37) *x*, *x* and *y* are collinear and *x*, *y* and *y* are collinear and *x*, *y* and *x* are collinear.

(38) Suppose $x \neq y$ and *x*, *y* and *z* are collinear and *x*, *y* and *t* are collinear and *x*, *y* and *u* are collinear. Then *z*, *t* and *u* are collinear.

(39) If $x \neq y$ and *x*, *y* and *z* are collinear and $x, y \parallel z, t$, then *x*, *y* and *t* are collinear.

(40) If *x*, *y* and *z* are collinear and *x*, *y* and *t* are collinear, then $x, y \parallel z, t$.

(41) Suppose $u \neq z$ and *x*, *y* and *u* are collinear and *x*, *y* and *z* are collinear and *u*, *z* and *w* are collinear. Then *x*, *y* and *w* are collinear.

(42) There exist *x*, *y*, *z* such that *x*, *y* and *z* are not collinear.

(43) If $x \neq y$, then there exists *z* such that *x*, *y* and *z* are not collinear.

In the sequel A_1 is a non empty affine structure.

Next we state two propositions:

(45)[5] Suppose $A_1 = \Lambda(S)$. Let *a*, *b*, *c*, *d* be elements of *S* and a', b', c', d' be elements of A_1. If $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$, then $a', b' \upuparrows c', d'$ iff $a, b \parallel c, d$.

(46) Suppose $A_1 = \Lambda(S)$. Then

(i) there exist elements *x*, *y* of A_1 such that $x \neq y$,

(ii) for all elements *x*, *y*, *z*, *t*, *u*, *w* of A_1 holds $x, y \upuparrows y, x$ and $x, y \upuparrows z, z$ and if $x \neq y$ and $x, y \upuparrows z, t$ and $x, y \upuparrows u, w$, then $z, t \upuparrows u, w$ and if $x, y \upuparrows x, z$, then $y, x \upuparrows y, z$,

(iii) there exist elements *x*, *y*, *z* of A_1 such that $x, y \not\upuparrows x, z$,

(iv) for all elements *x*, *y*, *z* of A_1 there exists an element *t* of A_1 such that $x, z \upuparrows y, t$ and $y \neq t$,

(v) for all elements *x*, *y*, *z* of A_1 there exists an element *t* of A_1 such that $x, y \upuparrows z, t$ and $x, z \upuparrows y, t$, and

(vi) for all elements *x*, *y*, *z*, *t* of A_1 such that $z, x \upuparrows x, t$ and $x \neq z$ there exists an element *u* of A_1 such that $y, x \upuparrows x, u$ and $y, z \upuparrows t, u$.

Let I_1 be a non empty affine structure. We say that I_1 is affine space-like if and only if the conditions (Def. 7) are satisfied.

[4] The proposition (33) has been removed.
[5] The proposition (44) has been removed.

(Def. 7)[6] (i) For all elements x, y, z, t, u, w of I_1 holds $x,y \upharpoonleft\upharpoonright y,x$ and $x,y \upharpoonleft\upharpoonright z,z$ and if $x \neq y$ and $x,y \upharpoonleft\upharpoonright z,t$ and $x,y \upharpoonleft\upharpoonright u,w$, then $z,t \upharpoonleft\upharpoonright u,w$ and if $x,y \upharpoonleft\upharpoonright x,z$, then $y,x \upharpoonleft\upharpoonright y,z$,

(ii) there exist elements x, y, z of I_1 such that $x,y \not\upharpoonleft\upharpoonright x,z$,

(iii) for all elements x, y, z of I_1 there exists an element t of I_1 such that $x,z \upharpoonleft\upharpoonright y,t$ and $y \neq t$,

(iv) for all elements x, y, z of I_1 there exists an element t of I_1 such that $x,y \upharpoonleft\upharpoonright z,t$ and $x,z \upharpoonleft\upharpoonright y,t$, and

(v) for all elements x, y, z, t of I_1 such that $z,x \upharpoonleft\upharpoonright x,t$ and $x \neq z$ there exists an element u of I_1 such that $y,x \upharpoonleft\upharpoonright x,u$ and $y,z \upharpoonleft\upharpoonright t,u$.

One can check that there exists a non empty affine structure which is strict, non trivial, and affine space-like.

An affine space is a non trivial affine space-like non empty affine structure.

One can prove the following propositions:

(47) Let A_1 be an affine space. Then

(i) there exist elements x, y of A_1 such that $x \neq y$,

(ii) for all elements x, y, z, t, u, w of A_1 holds $x,y \upharpoonleft\upharpoonright y,x$ and $x,y \upharpoonleft\upharpoonright z,z$ and if $x \neq y$ and $x,y \upharpoonleft\upharpoonright z,t$ and $x,y \upharpoonleft\upharpoonright u,w$, then $z,t \upharpoonleft\upharpoonright u,w$ and if $x,y \upharpoonleft\upharpoonright x,z$, then $y,x \upharpoonleft\upharpoonright y,z$,

(iii) there exist elements x, y, z of A_1 such that $x,y \not\upharpoonleft\upharpoonright x,z$,

(iv) for all elements x, y, z of A_1 there exists an element t of A_1 such that $x,z \upharpoonleft\upharpoonright y,t$ and $y \neq t$,

(v) for all elements x, y, z of A_1 there exists an element t of A_1 such that $x,y \upharpoonleft\upharpoonright z,t$ and $x,z \upharpoonleft\upharpoonright y,t$, and

(vi) for all elements x, y, z, t of A_1 such that $z,x \upharpoonleft\upharpoonright x,t$ and $x \neq z$ there exists an element u of A_1 such that $y,x \upharpoonleft\upharpoonright x,u$ and $y,z \upharpoonleft\upharpoonright t,u$.

(48) $\Lambda(S)$ is an affine space.

(49) There exist elements x, y of A_1 such that $x \neq y$ and for all elements x, y, z, t, u, w of A_1 holds $x,y \upharpoonleft\upharpoonright y,x$ and $x,y \upharpoonleft\upharpoonright z,z$ and if $x \neq y$ and $x,y \upharpoonleft\upharpoonright z,t$ and $x,y \upharpoonleft\upharpoonright u,w$, then $z,t \upharpoonleft\upharpoonright u,w$ and if $x,y \upharpoonleft\upharpoonright x,z$, then $y,x \upharpoonleft\upharpoonright y,z$ and there exist elements x, y, z of A_1 such that $x,y \not\upharpoonleft\upharpoonright x,z$ and for all elements x, y, z of A_1 there exists an element t of A_1 such that $x,z \upharpoonleft\upharpoonright y,t$ and $y \neq t$ and for all elements x, y, z of A_1 there exists an element t of A_1 such that $x,y \upharpoonleft\upharpoonright z,t$ and $x,z \upharpoonleft\upharpoonright y,t$ and for all elements x, y, z, t of A_1 such that $z,x \upharpoonleft\upharpoonright x,t$ and $x \neq z$ there exists an element u of A_1 such that $y,x \upharpoonleft\upharpoonright x,u$ and $y,z \upharpoonleft\upharpoonright t,u$ if and only if A_1 is an affine space.

We adopt the following convention: S denotes an ordered affine plane and x, y, z, t, u denote elements of S.

We now state two propositions:

(50) If $x,y \nparallel z,t$, then there exists u such that $x,y \parallel x,u$ and $z,t \parallel z,u$.

(51) Suppose $A_1 = \Lambda(S)$. Let x, y, z, t be elements of A_1. If $x,y \not\upharpoonleft\upharpoonright z,t$, then there exists an element u of A_1 such that $x,y \upharpoonleft\upharpoonright x,u$ and $z,t \upharpoonleft\upharpoonright z,u$.

Let I_1 be a non empty affine structure. We say that I_1 is 2-dimensional if and only if:

(Def. 8) For all elements x, y, z, t of I_1 such that $x,y \not\upharpoonleft\upharpoonright z,t$ there exists an element u of I_1 such that $x,y \upharpoonleft\upharpoonright x,u$ and $z,t \upharpoonleft\upharpoonright z,u$.

Let us mention that there exists an affine space which is strict and 2-dimensional.

An affine plane is a 2-dimensional affine space.

The following two propositions are true:

(53)[7] $\Lambda(S)$ is an affine plane.

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[6] The definition (Def. 6) has been removed.

[7] The proposition (52) has been removed.

(54) A_1 is an affine plane if and only if the following conditions are satisfied:

there exist elements x, y of A_1 such that $x \neq y$ and for all elements x, y, z, t, u, w of A_1 holds $x,y \upuparrows y,x$ and $x,y \upuparrows z,z$ and if $x \neq y$ and $x,y \upuparrows z,t$ and $x,y \upuparrows u,w$, then $z,t \upuparrows u,w$ and if $x,y \upuparrows x,z$, then $y,x \upuparrows y,z$ and there exist elements x, y, z of A_1 such that $x,y \not\upuparrows x,z$ and for all elements x, y, z of A_1 there exists an element t of A_1 such that $x,z \upuparrows y,t$ and $y \neq t$ and for all elements x, y, z of A_1 there exists an element t of A_1 such that $x,y \upuparrows z,t$ and $x,z \upuparrows y,t$ and for all elements x, y, z, t of A_1 such that $z,x \upuparrows x,t$ and $x \neq z$ there exists an element u of A_1 such that $y,x \upuparrows x,u$ and $y,z \upuparrows t,u$ and for all elements x, y, z, t of A_1 such that $x,y \not\upuparrows z,t$ there exists an element u of A_1 such that $x,y \upuparrows x,u$ and $z,t \upuparrows z,u$.

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