JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Locally Connected Spaces

Beata Padlewska
Technical University of Białystok

Summary. This article is a continuation of [3]. We define a neighbourhood of a point and a neighbourhood of a set and prove some facts about them. Then the definitions of a locally connected space and a locally connected set are introduced. Some theorems about locally connected spaces are given (based on [2]). We also define a quasi-component of a point and prove some of its basic properties.

MML Identifier: CONNSP_2.

WWW: http://mizar.org/JFM/Vol2/connsp_2.html

The articles [5], [6], [4], [7], [3], and [1] provide the notation and terminology for this paper.

Let X be a non empty topological space and let x be a point of X. A subset of X is called a neighbourhood of x if:

(Def. 1) $x \in \mathrm{Int}\,\mathrm{it}$.

Let X be a non empty topological space and let A be a subset of X. A subset of X is called a neighbourhood of A if:

(Def. 2) $A \subseteq \mathrm{Int}\,\mathrm{it}$.

In the sequel X denotes a non empty topological space, x denotes a point of X, and U_1 denotes a subset of X.

The following propositions are true:

(3)[1] Let A, B be subsets of X. Suppose A is a neighbourhood of x and B is a neighbourhood of x. Then $A \cup B$ is a neighbourhood of x.

(4) Let A, B be subsets of X. Then A is a neighbourhood of x and B is a neighbourhood of x if and only if $A \cap B$ is a neighbourhood of x.

(5) For every subset U_1 of X and for every point x of X such that U_1 is open and $x \in U_1$ holds U_1 is a neighbourhood of x.

(6) For every subset U_1 of X and for every point x of X such that U_1 is a neighbourhood of x holds $x \in U_1$.

(7) Suppose U_1 is a neighbourhood of x. Then there exists a non empty subset V of X such that V is a neighbourhood of x and open and $V \subseteq U_1$.

(8) U_1 is a neighbourhood of x iff there exists a subset V of X such that V is open and $V \subseteq U_1$ and $x \in V$.

[1] The propositions (1) and (2) have been removed.

(9) Let U_1 be a subset of X. Then U_1 is open if and only if for every x such that $x \in U_1$ there exists a subset A of X such that A is a neighbourhood of x and $A \subseteq U_1$.

(10) For every subset V of X holds V is a neighbourhood of $\{x\}$ iff V is a neighbourhood of x.

(11) Let B be a non empty subset of X, x be a point of $X \restriction B$, A be a subset of $X \restriction B$, A_1 be a subset of X, and x_1 be a point of X. Suppose B is open and A is a neighbourhood of x and $A = A_1$ and $x = x_1$. Then A_1 is a neighbourhood of x_1.

(12) Let B be a non empty subset of X, x be a point of $X \restriction B$, A be a subset of $X \restriction B$, A_1 be a subset of X, and x_1 be a point of X. Suppose A_1 is a neighbourhood of x_1 and $A = A_1$ and $x = x_1$. Then A is a neighbourhood of x.

(13) Let A be a subset of X and B be a subset of X. If A is a component of X and $A \subseteq B$, then A is a component of B.

(14) For every non empty subspace X_1 of X and for every point x of X and for every point x_1 of X_1 such that $x = x_1$ holds $\mathrm{Component}(x_1) \subseteq \mathrm{Component}(x)$.

Let X be a non empty topological space and let x be a point of X. We say that X is locally connected in x if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let U_1 be a subset of X. Suppose U_1 is a neighbourhood of x. Then there exists a subset V of X such that V is a neighbourhood of x and connected and $V \subseteq U_1$.

Let X be a non empty topological space. We say that X is locally connected if and only if:

(Def. 4) For every point x of X holds X is locally connected in x.

Let X be a non empty topological space, let A be a subset of X, and let x be a point of X. We say that A is locally connected in x if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let B be a non empty subset of X. Suppose $A = B$. Then there exists a point x_1 of $X \restriction B$ such that $x_1 = x$ and $X \restriction B$ is locally connected in x_1.

Let X be a non empty topological space and let A be a non empty subset of X. We say that A is locally connected if and only if:

(Def. 6) $X \restriction A$ is locally connected.

We now state a number of propositions:

(19)[2] Let given x. Then X is locally connected in x if and only if for all subsets V, S of X such that V is a neighbourhood of x and S is a component of V and $x \in S$ holds S is a neighbourhood of x.

(20) Let given x. Then X is locally connected in x if and only if for every non empty subset U_1 of X such that U_1 is open and $x \in U_1$ there exists a point x_1 of $X \restriction U_1$ such that $x_1 = x$ and $x \in \mathrm{Int}\,\mathrm{Component}(x_1)$.

(21) If X is locally connected, then for every subset S of X such that S is a component of X holds S is open.

(22) Suppose X is locally connected in x. Let A be a non empty subset of X. If A is open and $x \in A$, then A is locally connected in x.

(23) If X is locally connected, then for every non empty subset A of X such that A is open holds A is locally connected.

(24) X is locally connected if and only if for every non empty subset A of X and for every subset B of X such that A is open and B is a component of A holds B is open.

[2] The propositions (15)–(18) have been removed.

(25) Suppose X is locally connected. Let E be a non empty subset of X and C be a non empty subset of $X \restriction E$. Suppose C is connected and open. Then there exists a subset H of X such that H is open and connected and $C = E \cap H$.

(26) X is a T_4 space if and only if for all subsets A, C of X such that $A \neq \emptyset$ and $C \neq \Omega_X$ and $A \subseteq C$ and A is closed and C is open there exists a subset G of X such that G is open and $A \subseteq G$ and $\overline{G} \subseteq C$.

(27) Suppose X is locally connected and a T_4 space. Let A, B be subsets of X. Suppose $A \neq \emptyset$ and $B \neq \emptyset$ and A is closed and B is closed and A misses B. Suppose A is connected and B is connected. Then there exist subsets R, S of X such that R is connected and S is connected and R is open and S is open and $A \subseteq R$ and $B \subseteq S$ and R misses S.

(28) Let x be a point of X and F be a family of subsets of X. Suppose that for every subset A of X holds $A \in F$ iff A is open and closed and $x \in A$. Then $F \neq \emptyset$.

Let X be a non empty topological space and let x be a point of X. The quasi-component of x is a subset of X and is defined by the condition (Def. 7).

(Def. 7) There exists a family F of subsets of X such that

(i) for every subset A of X holds $A \in F$ iff A is open and closed and $x \in A$, and

(ii) $\bigcap F =$ the quasi-component of x.

The following propositions are true:

(30)[3] $x \in$ the quasi-component of x.

(31) Let A be a subset of X. Suppose A is open and closed and $x \in A$. Suppose $A \subseteq$ the quasi-component of x. Then $A =$ the quasi-component of x.

(32) The quasi-component of x is closed.

(33) Component$(x) \subseteq$ the quasi-component of x.

REFERENCES

[1] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/compts_1.html`.

[2] Kazimierz Kuratowski. *Wstęp do teorii mnogości i topologii*. PWN, Warszawa, 1977.

[3] Beata Padlewska. Connected spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/connsp_1.html`.

[4] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[6] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[7] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_1.html`.

Received September 5, 1990

Published January 2, 2004

[3] The proposition (29) has been removed.