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Connected Spaces

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Summary. The following notions are defined: separated sets, connected spaces, connected sets, components of a topological space, the component of a point. The definition of the boundary of a set is also included. The singleton of a point of a topological space is redefined as a subset of the space. Some theorems about these notions are proved.

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The articles [3], [4], [1], [2], and [5] provide the notation and terminology for this paper.

For simplicity, we use the following convention: G_1 is a topological space, A, B, C are subsets of G_1, T_1 is a topological structure, and K, K_1, L, L_1 are subsets of T_1.

Let G_1 be a topological structure and let A, B be subsets of G_1. We say that A and B are separated if and only if:

(Def. 1) $\overline{A}$ misses B and A misses $\overline{B}$.

Let us note that the predicate A and B are separated is symmetric.

We now state several propositions:

(2)[1] If K and L are separated, then K misses L.

(3) If $\Omega_{(T_1)} = K \cup L$ and K is closed and L is closed and K misses L, then K and L are separated.

(4) If $\Omega_{(G_1)} = A \cup B$ and A is open and B is open and A misses B, then A and B are separated.

(5) If $\Omega_{(G_1)} = A \cup B$ and A and B are separated, then A is open and closed and B is open and closed.

(6) Let X' be a subspace of G_1, P_1, Q_1 be subsets of G_1, and P, Q be subsets of X'. Suppose $P = P_1$ and $Q = Q_1$. If P and Q are separated, then P_1 and Q_1 are separated.

(7) Let X' be a subspace of G_1, P, Q be subsets of G_1, and P_1, Q_1 be subsets of X'. Suppose $P = P_1$ and $Q = Q_1$ and $P \cup Q \subseteq \Omega_{X'}$. If P and Q are separated, then P_1 and Q_1 are separated.

(8) If K and L are separated and $K_1 \subseteq K$ and $L_1 \subseteq L$, then K_1 and L_1 are separated.

(9) If A and B are separated and A and C are separated, then A and $B \cup C$ are separated.

(10) If K is closed and L is closed or K is open and L is open, then $K \setminus L$ and $L \setminus K$ are separated.

Let G_1 be a topological structure. We say that G_1 is connected if and only if:

[1] The proposition (1) has been removed.

(Def. 2) For all subsets A, B of G_1 such that $\Omega_{(G_1)} = A \cup B$ and A and B are separated holds $A = \emptyset_{(G_1)}$ or $B = \emptyset_{(G_1)}$.

Next we state several propositions:

(11) G_1 is connected if and only if for all subsets A, B of G_1 such that $\Omega_{(G_1)} = A \cup B$ and $A \neq \emptyset_{(G_1)}$ and $B \neq \emptyset_{(G_1)}$ and A is closed and B is closed holds A meets B.

(12) G_1 is connected if and only if for all subsets A, B of G_1 such that $\Omega_{(G_1)} = A \cup B$ and $A \neq \emptyset_{(G_1)}$ and $B \neq \emptyset_{(G_1)}$ and A is open and B is open holds A meets B.

(13) G_1 is connected iff for every subset A of G_1 such that $A \neq \emptyset_{(G_1)}$ and $A \neq \Omega_{(G_1)}$ holds $\overline{A}$ meets $\overline{\Omega_{(G_1)} \setminus A}$.

(14) G_1 is connected iff for every subset A of G_1 such that A is open and closed holds $A = \emptyset_{(G_1)}$ or $A = \Omega_{(G_1)}$.

(15) Let G_2 be a topological space and F be a map from G_1 into G_2. If F is continuous and $F^{\circ}(\Omega_{(G_1)}) = \Omega_{(G_2)}$ and G_1 is connected, then G_2 is connected.

Let G_1 be a topological structure and let A be a subset of G_1. We say that A is connected if and only if:

(Def. 3) $G_1 \restriction A$ is connected.

The following propositions are true:

(16) A is connected if and only if for all subsets P, Q of G_1 such that $A = P \cup Q$ and P and Q are separated holds $P = \emptyset_{(G_1)}$ or $Q = \emptyset_{(G_1)}$.

(17) If A is connected and $A \subseteq B \cup C$ and B and C are separated, then $A \subseteq B$ or $A \subseteq C$.

(18) If A is connected and B is connected and A and B are not separated, then $A \cup B$ is connected.

(19) If C is connected and $C \subseteq A$ and $A \subseteq \overline{C}$, then A is connected.

(20) If A is connected, then $\overline{A}$ is connected.

(21) Suppose G_1 is connected and A is connected and $\Omega_{(G_1)} \setminus A = B \cup C$ and B and C are separated. Then $A \cup B$ is connected and $A \cup C$ is connected.

(22) If $\Omega_{(G_1)} \setminus A = B \cup C$ and B and C are separated and A is closed, then $A \cup B$ is closed and $A \cup C$ is closed.

(23) If C is connected and C meets A and $C \setminus A \neq \emptyset_{(G_1)}$, then C meets $\mathrm{Fr} A$.

(24) Let X' be a subspace of G_1, A be a subset of G_1, and B be a subset of X'. If $A = B$, then A is connected iff B is connected.

(25) If A is closed and B is closed and $A \cup B$ is connected and $A \cap B$ is connected, then A is connected and B is connected.

(26) Let F be a family of subsets of G_1. Suppose that

(i) for every subset A of G_1 such that $A \in F$ holds A is connected, and

(ii) there exists a subset A of G_1 such that $A \neq \emptyset_{(G_1)}$ and $A \in F$ and for every subset B of G_1 such that $B \in F$ and $B \neq A$ holds A and B are not separated.

Then $\bigcup F$ is connected.

(27) Let F be a family of subsets of G_1. Suppose for every subset A of G_1 such that $A \in F$ holds A is connected and $\bigcap F \neq \emptyset_{(G_1)}$. Then $\bigcup F$ is connected.

(28) $\Omega_{(G_1)}$ is connected iff G_1 is connected.

(29) For every non empty topological space G_1 and for every point x of G_1 holds $\{x\}$ is connected.

Let G_1 be a topological structure and let x, y be points of G_1. We say that x and y are joined if and only if:

(Def. 4) There exists a subset C of G_1 such that C is connected and $x \in C$ and $y \in C$.

The following four propositions are true:

(30) Let G_1 be a non empty topological space. Given a point x of G_1 such that let y be a point of G_1. Then x and y are joined. Then G_1 is connected.

(31) There exists a point x of G_1 such that for every point y of G_1 holds x and y are joined if and only if for all points x, y of G_1 holds x and y are joined.

(32) Let G_1 be a non empty topological space. Suppose that for all points x, y of G_1 holds x and y are joined. Then G_1 is connected.

(33) Let G_1 be a non empty topological space, x be a point of G_1, and F be a family of subsets of G_1. Suppose that for every subset A of G_1 holds $A \in F$ iff A is connected and $x \in A$. Then $F \neq \emptyset$.

Let G_1 be a topological structure and let A be a subset of G_1. We say that A is a component of G_1 if and only if:

(Def. 5) A is connected and for every subset B of G_1 such that B is connected holds if $A \subseteq B$, then $A = B$.

We now state several propositions:

(34) For every non empty topological space G_1 and for every subset A of G_1 such that A is a component of G_1 holds $A \neq \emptyset_{(G_1)}$.

(35) If A is a component of G_1, then A is closed.

(36) If A is a component of G_1 and B is a component of G_1, then $A = B$ or A and B are separated.

(37) If A is a component of G_1 and B is a component of G_1, then $A = B$ or A misses B.

(38) If C is connected, then for every subset S of G_1 such that S is a component of G_1 holds C misses S or $C \subseteq S$.

Let G_1 be a topological structure and let A, B be subsets of G_1. We say that B is a component of A if and only if:

(Def. 6) There exists a subset B_1 of $G_1 \restriction A$ such that $B_1 = B$ and B_1 is a component of $G_1 \restriction A$.

The following proposition is true

(39) If G_1 is connected and A is connected and C is a component of $\Omega_{(G_1)} \setminus A$, then $\Omega_{(G_1)} \setminus C$ is connected.

Let G_1 be a topological structure and let x be a point of G_1. The functor $\mathrm{Component}(x)$ yielding a subset of G_1 is defined by the condition (Def. 7).

(Def. 7) There exists a family F of subsets of G_1 such that for every subset A of G_1 holds $A \in F$ iff A is connected and $x \in A$ and $\bigcup F = \mathrm{Component}(x)$.

In the sequel G_1 denotes a non empty topological space, A, C denote subsets of G_1, and x denotes a point of G_1.

We now state several propositions:

(40) $x \in \mathrm{Component}(x)$.

(41) $\mathrm{Component}(x)$ is connected.

(42) If C is connected and $\mathrm{Component}(x) \subseteq C$, then $C = \mathrm{Component}(x)$.

(43) A is a component of G_1 iff there exists a point x of G_1 such that $A = \mathrm{Component}(x)$.

(44) If A is a component of G_1 and $x \in A$, then $A = \mathrm{Component}(x)$.

(45) If $A = \mathrm{Component}(x)$, then for every point p of G_1 such that $p \in A$ holds $\mathrm{Component}(p) = A$.

(46) Let F be a family of subsets of G_1. Suppose that for every subset A of G_1 holds $A \in F$ iff A is a component of G_1. Then F is a cover of G_1.

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