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Compact Spaces

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Summary. The article contains definition of a compact space and some theorems about compact spaces. The notions of a cover of a set and a centered family are defined in the article to be used in these theorems. A set is compact in the topological space if and only if every open cover of the set has a finite subcover. This definition is equivalent, what has been shown next, to the following definition: a set is compact if and only if a subspace generated by that set is compact. Some theorems about mappings and homeomorphisms of compact spaces have been also proved. The following schemes used in proofs of theorems have been proved in the article: FuncExChoice – the scheme of choice of a function, BiFuncEx – the scheme of parallel choice of two functions and the theorem about choice of a finite counter image of a finite image.

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The articles [8], [3], [9], [10], [1], [2], [6], [5], [7], and [4] provide the notation and terminology for this paper.

We adopt the following rules: x, y, z, Y, Z denote sets and f denotes a function.

In this article we present several logical schemes. The scheme *NonUniqBoundFuncEx* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{rng} f \subseteq \mathcal{B}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters meet the following condition:

- For every x such that $x \in \mathcal{A}$ there exists y such that $y \in \mathcal{B}$ and $\mathcal{P}[x,y]$.

The scheme *BiFuncEx* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exist functions f, g such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{dom} g = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x), g(x)]$

provided the parameters satisfy the following condition:

- If $x \in \mathcal{A}$, then there exist y, z such that $y \in \mathcal{B}$ and $z \in \mathcal{C}$ and $\mathcal{P}[x,y,z]$.

We now state the proposition

(1) If Z is finite and $Z \subseteq \operatorname{rng} f$, then there exists Y such that $Y \subseteq \operatorname{dom} f$ and Y is finite and $f^{\circ}Y = Z$.

In the sequel T is a topological structure, A is a subspace of T, and P, Q are subsets of T.

Let T be a 1-sorted structure, let F be a family of subsets of T, and let P be a subset of T. We say that F is a cover of P if and only if:

(Def. 1) $P \subseteq \bigcup F$.

Let F be a set. We say that F is centered if and only if:

(Def. 2) $F \neq \emptyset$ and for every set G such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite holds $\bigcap G \neq \emptyset$.

Let T be a topological structure. We say that T is compact if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let F be a family of subsets of T. Suppose F is a cover of T and open. Then there exists a family G of subsets of T such that $G \subseteq F$ and G is a cover of T and finite.

We say that T is T_2 if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let p, q be points of T. Suppose $p \neq q$. Then there exist subsets W, V of T such that W is open and V is open and $p \in W$ and $q \in V$ and W misses V.

We introduce T is a T_2 space as a synonym of T is T_2. We say that T is T_3 if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let p be a point of T and P be a subset of T. Suppose $P \neq \emptyset$ and P is closed and $p \notin P$. Then there exist subsets W, V of T such that W is open and V is open and $p \in W$ and $P \subseteq V$ and W misses V.

We introduce T is a T_3 space as a synonym of T is T_3. We say that T is T_4 if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let W, V be subsets of T. Suppose $W \neq \emptyset$ and $V \neq \emptyset$ and W is closed and V is closed and W misses V. Then there exist subsets P, Q of T such that P is open and Q is open and $W \subseteq P$ and $V \subseteq Q$ and P misses Q.

We introduce T is a T_4 space as a synonym of T is T_4.

Let T be a topological structure and let P be a subset of T. We say that P is compact if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let F be a family of subsets of T. Suppose F is a cover of P and open. Then there exists a family G of subsets of T such that $G \subseteq F$ and G is a cover of P and finite.

One can prove the following propositions:

(9)[1] $\emptyset_T$ is compact.

(10) T is compact iff Ω_T is compact.

(11) If $Q \subseteq \Omega_A$, then Q is compact iff for every subset P of A such that $P = Q$ holds P is compact.

(12)(i) If $P = \emptyset$, then P is compact iff $T \restriction P$ is compact, and

(ii) if T is topological space-like and $P \neq \emptyset$, then P is compact iff $T \restriction P$ is compact.

(13) Let T be a non empty topological space. Then T is compact if and only if for every family F of subsets of T such that F is centered and closed holds $\bigcap F \neq \emptyset$.

(14) Let T be a non empty topological space. Then T is compact if and only if for every family F of subsets of T such that $F \neq \emptyset$ and F is closed and $\bigcap F = \emptyset$ there exists a family G of subsets of T such that $G \neq \emptyset$ and $G \subseteq F$ and G is finite and $\bigcap G = \emptyset$.

In the sequel T_1 is a topological space and P_1, Q_1 are subsets of T_1.

Next we state several propositions:

(15) Suppose T_1 is a T_2 space. Let A be a subset of T_1. Suppose $A \neq \emptyset$ and A is compact. Let p be a point of T_1. Suppose $p \notin A$. Then there exist P_1, Q_1 such that P_1 is open and Q_1 is open and $p \in P_1$ and $A \subseteq Q_1$ and P_1 misses Q_1.

(16) If T_1 is a T_2 space and P_1 is compact, then P_1 is closed.

[1] The propositions (2)–(8) have been removed.

(17) If T is compact and P is closed, then P is compact.

(18) If P_1 is compact and $Q_1 \subseteq P_1$ and Q_1 is closed, then Q_1 is compact.

(19) If P is compact and Q is compact, then $P \cup Q$ is compact.

(20) If T_1 is a T_2 space and P_1 is compact and Q_1 is compact, then $P_1 \cap Q_1$ is compact.

(21) If T_1 is a T_2 space and compact, then T_1 is a T_3 space.

(22) If T_1 is a T_2 space and compact, then T_1 is a T_4 space.

In the sequel S is a non empty topological structure and f is a map from T into S.
We now state two propositions:

(23) If T is compact and f is continuous and $\operatorname{rng} f = \Omega_S$, then S is compact.

(24) If f is continuous and $\operatorname{rng} f = \Omega_S$ and P is compact, then $f^\circ P$ is compact.

In the sequel S_1 is a non empty topological space and f is a map from T_1 into S_1.
The following two propositions are true:

(25) Suppose T_1 is compact and S_1 is a T_2 space and $\operatorname{rng} f = \Omega_{(S_1)}$ and f is continuous. Let given P_1. If P_1 is closed, then $f^\circ P_1$ is closed.

(26) Suppose T_1 is compact and S_1 is a T_2 space and $\operatorname{dom} f = \Omega_{(T_1)}$ and $\operatorname{rng} f = \Omega_{(S_1)}$ and f is one-to-one and continuous. Then f is a homeomorphism.

References

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[5] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[6] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[9] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[10] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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