Trigonometric Form of Complex Numbers

Robert Milewski

University of Białystok

The articles [16], [21], [1], [17], [6], [18], [12], [13], [19], [9], [2], [22], [3], [8], [14], [7], [4], [5], [15], [20], [23], [10], and [11] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *Regr without 0* concerns a unary predicate $\mathcal{P}$, and states that:

$$\mathcal{P}[1]$$

provided the parameters meet the following requirements:

- There exists a non empty natural number k such that $\mathcal{P}[k]$, and
- For every non empty natural number k such that $k \neq 1$ and $\mathcal{P}[k]$ there exists a non empty natural number n such that $n < k$ and $\mathcal{P}[n]$.

The following propositions are true:

(3)[1]   For every element z of $\mathbb{C}$ holds $\Re(z) \geq -|z|$.

(4)   For every element z of $\mathbb{C}$ holds $\Im(z) \geq -|z|$.

(5)   For every element z of $\mathbb{C}_\mathrm{F}$ holds $\Re(z) \geq -|z|$.

(6)   For every element z of $\mathbb{C}_\mathrm{F}$ holds $\Im(z) \geq -|z|$.

(7)   For every element z of $\mathbb{C}_\mathrm{F}$ holds $|z|^2 = \Re(z)^2 + \Im(z)^2$.

(8)   For all real numbers x_1, x_2, y_1, y_2 such that $x_1 + x_2 i_{\mathbb{C}_\mathrm{F}} = y_1 + y_2 i_{\mathbb{C}_\mathrm{F}}$ holds $x_1 = y_1$ and $x_2 = y_2$.

(9)   For every element z of $\mathbb{C}_\mathrm{F}$ holds $z = \Re(z) + \Im(z) i_{\mathbb{C}_\mathrm{F}}$.

(10) $0_{\mathbb{C}_\mathrm{F}} = 0 + 0 i_{\mathbb{C}_\mathrm{F}}$.

(12)[2]   For every unital non empty groupoid L and for every element x of L holds $\mathrm{power}_L(x, 1) = x$.

(13)   For every unital non empty groupoid L and for every element x of L holds $\mathrm{power}_L(x, 2) = x \cdot x$.

(14)   Let L be an add-associative right zeroed right complementable right distributive unital non empty double loop structure and n be a natural number. If $n > 0$, then $\mathrm{power}_L(0_L, n) = 0_L$.

---

[1] The propositions (1) and (2) have been removed.

[2] The proposition (11) has been removed.

(15) Let L be an associative commutative unital non empty groupoid, x, y be elements of L, and n be a natural number. Then $\text{power}_L(x \cdot y, n) = \text{power}_L(x, n) \cdot \text{power}_L(y, n)$.

(16) For every real number x such that $x > 0$ and for every natural number n holds $\text{power}_{\mathbb{C}_F}(x + 0i_{\mathbb{C}_F}, n) = x^n + 0i_{\mathbb{C}_F}$.

(17) For every real number x and for every natural number n such that $x \geq 0$ and $n \neq 0$ holds $\sqrt[n]{x^n} = x$.

2. SINUS AND COSINUS PROPERTIES

Next we state several propositions:

(20)[3] $\pi + \frac{\pi}{2} = \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi + \frac{\pi}{2} = 2 \cdot \pi$ and $\frac{3}{2} \cdot \pi - \pi = \frac{\pi}{2}$.

(21) $0 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \pi$ and $0 < \pi$ and $-\frac{\pi}{2} < \frac{\pi}{2}$ and $\pi < 2 \cdot \pi$ and $\frac{\pi}{2} < \frac{3}{2} \cdot \pi$ and $-\frac{\pi}{2} < 0$ and $0 < 2 \cdot \pi$ and $\pi < \frac{3}{2} \cdot \pi$ and $\frac{3}{2} \cdot \pi < 2 \cdot \pi$ and $0 < \frac{3}{2} \cdot \pi$.

(22) For all real numbers a, b, c, x such that $x \in]a, c[$ holds $x \in]a, b[$ or $x = b$ or $x \in]b, c[$.

(23) For every real number x such that $x \in]0, \pi[$ holds $\sin(x) > 0$.

(24) For every real number x such that $x \in [0, \pi]$ holds $\sin(x) \geq 0$.

(25) For every real number x such that $x \in]\pi, 2 \cdot \pi[$ holds $\sin(x) < 0$.

(26) For every real number x such that $x \in [\pi, 2 \cdot \pi]$ holds $\sin(x) \leq 0$.

In the sequel x is a real number.
One can prove the following propositions:

(27) If $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, then $\cos(x) > 0$.

(28) For every real number x such that $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ holds $\cos(x) \geq 0$.

(29) If $x \in]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$, then $\cos(x) < 0$.

(30) For every real number x such that $x \in [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ holds $\cos(x) \leq 0$.

(31) If $x \in]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$, then $\cos(x) > 0$.

(32) For every real number x such that $x \in [\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ holds $\cos(x) \geq 0$.

(33) For every real number x such that $0 \leq x$ and $x < 2 \cdot \pi$ and $\sin x = 0$ holds $x = 0$ or $x = \pi$.

(34) For every real number x such that $0 \leq x$ and $x < 2 \cdot \pi$ and $\cos x = 0$ holds $x = \frac{\pi}{2}$ or $x = \frac{3}{2} \cdot \pi$.

(35) $\sin$ is increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(36) $\sin$ is decreasing on $]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$.

(37) $\cos$ is decreasing on $]0, \pi[$.

(38) $\cos$ is increasing on $]\pi, 2 \cdot \pi[$.

(39) $\sin$ is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(40) $\sin$ is decreasing on $[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$.

(41) $\cos$ is decreasing on $[0, \pi]$.

(42) $\cos$ is increasing on $[\pi, 2 \cdot \pi]$.

---

[3] The propositions (18) and (19) have been removed.

(43) sin is continuous on $\mathbb{R}$ and for all real numbers x, y holds sin is continuous on $[x,y]$ and sin is continuous on $]x,y[$.

(44) cos is continuous on $\mathbb{R}$ and for all real numbers x, y holds cos is continuous on $[x,y]$ and cos is continuous on $]x,y[$.

(45) $\sin(x) \in [-1,1]$ and $\cos(x) \in [-1,1]$.

(46) $\operatorname{rng} \sin = [-1,1]$.

(47) $\operatorname{rng} \cos = [-1,1]$.

(48) $\operatorname{rng}(\sin \restriction [-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1,1]$.

(49) $\operatorname{rng}(\sin \restriction [\frac{\pi}{2}, \frac{3}{2} \cdot \pi]) = [-1,1]$.

(50) $\operatorname{rng}(\cos \restriction [0, \pi]) = [-1,1]$.

(51) $\operatorname{rng}(\cos \restriction [\pi, 2 \cdot \pi]) = [-1,1]$.

3. ARGUMENT OF COMPLEX NUMBER

Let z be an element of $\mathbb{C}_{\mathrm{F}}$. The functor $\operatorname{Arg} z$ yields a real number and is defined by:

(Def. 1)(i) $z = |z| \cdot \cos \operatorname{Arg} z + (|z| \cdot \sin \operatorname{Arg} z) i_{\mathbb{C}_{\mathrm{F}}}$ and $0 \le \operatorname{Arg} z$ and $\operatorname{Arg} z < 2 \cdot \pi$ if $z \ne 0_{\mathbb{C}_{\mathrm{F}}}$,

(ii) $\operatorname{Arg} z = 0$, otherwise.

One can prove the following propositions:

(52) For every element z of $\mathbb{C}_{\mathrm{F}}$ holds $0 \le \operatorname{Arg} z$ and $\operatorname{Arg} z < 2 \cdot \pi$.

(53) For every real number x such that $x \ge 0$ holds $\operatorname{Arg}(x + 0 i_{\mathbb{C}_{\mathrm{F}}}) = 0$.

(54) For every real number x such that $x < 0$ holds $\operatorname{Arg}(x + 0 i_{\mathbb{C}_{\mathrm{F}}}) = \pi$.

(55) For every real number x such that $x > 0$ holds $\operatorname{Arg}(0 + x i_{\mathbb{C}_{\mathrm{F}}}) = \frac{\pi}{2}$.

(56) For every real number x such that $x < 0$ holds $\operatorname{Arg}(0 + x i_{\mathbb{C}_{\mathrm{F}}}) = \frac{3}{2} \cdot \pi$.

(57) $\operatorname{Arg}(\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}) = 0$.

(58) $\operatorname{Arg} i_{\mathbb{C}_{\mathrm{F}}} = \frac{\pi}{2}$.

(59) For every element z of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(60) For every element z of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(61) For every element z of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]\pi, \frac{3}{2} \cdot \pi[$ iff $\Re(z) < 0$ and $\Im(z) < 0$.

(62) For every element z of $\mathbb{C}_{\mathrm{F}}$ holds $\operatorname{Arg} z \in]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ iff $\Re(z) > 0$ and $\Im(z) < 0$.

(63) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) > 0$ holds $\sin \operatorname{Arg} z > 0$.

(64) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) < 0$ holds $\sin \operatorname{Arg} z < 0$.

(65) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) \ge 0$ holds $\sin \operatorname{Arg} z \ge 0$.

(66) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(z) \le 0$ holds $\sin \operatorname{Arg} z \le 0$.

(67) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) > 0$ holds $\cos \operatorname{Arg} z > 0$.

(68) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) < 0$ holds $\cos \operatorname{Arg} z < 0$.

(69) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(z) \ge 0$ holds $\cos \operatorname{Arg} z \ge 0$.

(70) For every element z of $\mathbb{C}_F$ such that $\Re(z) \leq 0$ and $z \neq 0_{\mathbb{C}_F}$ holds $\cos \operatorname{Arg} z \leq 0$.

(71) For every natural number n holds $\operatorname{power}_{\mathbb{C}_F}(\cos x + \sin x i_{\mathbb{C}_F}, n) = \cos(n \cdot x) + \sin(n \cdot x) i_{\mathbb{C}_F}$.

(72) Let z be an element of $\mathbb{C}_F$ and n be a natural number. If $z \neq 0_{\mathbb{C}_F}$ or $n \neq 0$, then $\operatorname{power}_{\mathbb{C}_F}(z, n) = |z|^n \cdot \cos(n \cdot \operatorname{Arg} z) + (|z|^n \cdot \sin(n \cdot \operatorname{Arg} z)) i_{\mathbb{C}_F}$.

(73) For all natural numbers n, k such that $n \neq 0$ holds $\operatorname{power}_{\mathbb{C}_F}(\cos(\frac{x+2\cdot\pi\cdot k}{n}) + \sin(\frac{x+2\cdot\pi\cdot k}{n}) i_{\mathbb{C}_F}, n) = \cos x + \sin x i_{\mathbb{C}_F}$.

(74) Let z be an element of $\mathbb{C}_F$ and n, k be natural numbers. If $n \neq 0$, then $z = \operatorname{power}_{\mathbb{C}_F}(\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z + 2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z + 2\cdot\pi\cdot k}{n})) i_{\mathbb{C}_F}, n)$.

Let x be an element of $\mathbb{C}_F$ and let n be a non empty natural number. An element of $\mathbb{C}_F$ is called a complex root of n, x if:

(Def. 2) $\operatorname{power}_{\mathbb{C}_F}(\text{it}, n) = x$.

One can prove the following four propositions:

(75) Let x be an element of $\mathbb{C}_F$, n be a non empty natural number, and k be a natural number. Then $\sqrt[n]{|x|} \cdot \cos(\frac{\operatorname{Arg} x + 2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|x|} \cdot \sin(\frac{\operatorname{Arg} x + 2\cdot\pi\cdot k}{n})) i_{\mathbb{C}_F}$ is a complex root of n, x.

(76) For every element x of $\mathbb{C}_F$ and for every complex root v of $1, x$ holds $v = x$.

(77) For every non empty natural number n and for every complex root v of $n, 0_{\mathbb{C}_F}$ holds $v = 0_{\mathbb{C}_F}$.

(78) Let n be a non empty natural number, x be an element of $\mathbb{C}_F$, and v be a complex root of n, x. If $v = 0_{\mathbb{C}_F}$, then $x = 0_{\mathbb{C}_F}$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[5] Czesław Byliński. The complex numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/complex1.html.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[7] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[8] Jarosław Kotowicz. Properties of real functions. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rfunct_2.html.

[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[10] Anna Justyna Milewska. The field of complex numbers. *Journal of Formalized Mathematics*, 12, 2000. http://mizar.org/JFM/Vol12/complfld.html.

[11] Anna Justyna Milewska. The Hahn Banach theorem in the vector space over the field of complex numbers. *Journal of Formalized Mathematics*, 12, 2000. http://mizar.org/JFM/Vol12/hahnban1.html.

[12] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[13] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/power.html.

[14] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fcont_1.html.

[15] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rcomp_1.html`.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[17] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[19] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[20] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_1.html`.

[21] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[22] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[23] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Journal of Formalized Mathematics*, 10, 1998. `http://mizar.org/JFM/Vol10/sin_cos.html`.