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Inst. of Computer Science, Univ. of Białystok

The Field of Complex Numbers

Anna Justyna Milewska
University of Białystok

Summary. This article contains the definition and many facts about the field of complex numbers.

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The articles [9], [12], [1], [10], [5], [6], [8], [11], [7], [4], [3], and [2] provide the notation and terminology for this paper.

One can prove the following proposition

(2)[1] For all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ holds $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$.

The strict double loop structure $\mathbb{C}_{\rm F}$ is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\mathbb{C}_{\rm F} = \mathbb{C}$,

(ii) the addition of $\mathbb{C}_{\rm F} = +_{\mathbb{C}}$,

(iii) the multiplication of $\mathbb{C}_{\rm F} = \cdot_{\mathbb{C}}$,

(iv) the unity of $\mathbb{C}_{\rm F} = 1_{\mathbb{C}}$, and

(v) the zero of $\mathbb{C}_{\rm F} = 0_{\mathbb{C}}$.

Let us mention that $\mathbb{C}_{\rm F}$ is non empty.

Let us mention that every element of $\mathbb{C}_{\rm F}$ is complex.

One can verify that $\mathbb{C}_{\rm F}$ is add-associative, right zeroed, right complementable, Abelian, commutative, associative, left unital, right unital, distributive, field-like, and non degenerated.

Next we state several propositions:

(3) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 + y_1 = x_2 + y_2$.

(4) For every element x_1 of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ holds $-x_1 = -x_2$.

(5) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 - y_1 = x_2 - y_2$.

(6) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 \cdot y_1 = x_2 \cdot y_2$.

(7) For every element x_1 of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $x_1 \neq 0_{\mathbb{C}_{\rm F}}$ holds ${x_1}^{-1} = {x_2}^{-1}$.

[1] The proposition (1) has been removed.

(8) For all elements x_1, y_1 of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ and $y_1 \neq 0_{\mathbb{C}_{\rm F}}$ holds $\frac{x_1}{y_1} = \frac{x_2}{y_2}$.

(9) $0_{\mathbb{C}_{\rm F}} = 0_{\mathbb{C}}$.

(10) $\mathbf{1}_{\mathbb{C}_{\rm F}} = 1_{\mathbb{C}}$.

(11) $\mathbf{1}_{\mathbb{C}_{\rm F}} + \mathbf{1}_{\mathbb{C}_{\rm F}} \neq 0_{\mathbb{C}_{\rm F}}$.

Let z be an element of $\mathbb{C}_{\rm F}$. The functor $\overline{z}$ yielding an element of $\mathbb{C}_{\rm F}$ is defined as follows:

(Def. 2) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $\overline{z} = \overline{z'}$.

Let z be an element of $\mathbb{C}_{\rm F}$. The functor $|z|$ yields a real number and is defined as follows:

(Def. 3) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $|z| = |z'|$.

Let z be an element of $\mathbb{C}_{\rm F}$. Then $|z|$ is an element of $\mathbb{R}$.
The following proposition is true

(12) For every element x_1 of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ holds $\overline{x_1} = \overline{x_2}$.

In the sequel z, z_1, z_2, z_3, z_4 are elements of $\mathbb{C}_{\rm F}$.
Next we state a number of propositions:

(29)[2] $-z = (-\mathbf{1}_{\mathbb{C}_{\rm F}}) \cdot z$.

(35)[3] $z_1 - -z_2 = z_1 + z_2$.

(41)[4] $z_1 = (z_1 + z) - z$.

(42) $z_1 = (z_1 - z) + z$.

(47)[5] If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_1{}^{-1} = z_2{}^{-1}$, then $z_1 = z_2$.

(48) If $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and if $z_1 \cdot z_2 =$ the unity of $\mathbb{C}_{\rm F}$ or $z_2 \cdot z_1 =$ the unity of $\mathbb{C}_{\rm F}$, then $z_1 = z_2{}^{-1}$.

(49) If $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and if $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = z_3 \cdot z_2{}^{-1}$ and $z_1 = z_2{}^{-1} \cdot z_3$.

(50) (The unity of $\mathbb{C}_{\rm F})^{-1} =$ the unity of $\mathbb{C}_{\rm F}$.

(51) If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$, then $(z_1 \cdot z_2)^{-1} = z_1{}^{-1} \cdot z_2{}^{-1}$.

(53)[6] If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $(-z)^{-1} = -z^{-1}$.

(55)[7] If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$, then $z_1{}^{-1} + z_2{}^{-1} = (z_1 + z_2) \cdot (z_1 \cdot z_2)^{-1}$.

(56) If $z_1 \neq 0_{\mathbb{C}_{\rm F}}$ and $z_2 \neq 0_{\mathbb{C}_{\rm F}}$, then $z_1{}^{-1} - z_2{}^{-1} = (z_2 - z_1) \cdot (z_1 \cdot z_2)^{-1}$.

(58)[8] If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $z^{-1} = \frac{\text{the unity of } \mathbb{C}_{\rm F}}{z}$.

(59) $\frac{z}{\text{the unity of } \mathbb{C}_{\rm F}} = z$.

(60) If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $\frac{z}{z} =$ the unity of $\mathbb{C}_{\rm F}$.

(61) If $z \neq 0_{\mathbb{C}_{\rm F}}$, then $\frac{0_{\mathbb{C}_{\rm F}}}{z} = 0_{\mathbb{C}_{\rm F}}$.

(62) If $z_2 \neq 0_{\mathbb{C}_{\rm F}}$ and $\frac{z_1}{z_2} = 0_{\mathbb{C}_{\rm F}}$, then $z_1 = 0_{\mathbb{C}_{\rm F}}$.

[2] The propositions (13)–(28) have been removed.
[3] The propositions (30)–(34) have been removed.
[4] The propositions (36)–(40) have been removed.
[5] The propositions (43)–(46) have been removed.
[6] The proposition (52) has been removed.
[7] The proposition (54) has been removed.
[8] The proposition (57) has been removed.

(63) If $z_2 \neq 0_{\mathbb{C}_F}$ and $z_4 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 \cdot z_3}{z_2 \cdot z_4}$.

(64) If $z_2 \neq 0_{\mathbb{C}_F}$, then $z \cdot \frac{z_1}{z_2} = \frac{z \cdot z_1}{z_2}$.

(65) If $z_2 \neq 0_{\mathbb{C}_F}$ and $\frac{z_1}{z_2} =$ the unity of $\mathbb{C}_F$, then $z_1 = z_2$.

(66) If $z \neq 0_{\mathbb{C}_F}$, then $z_1 = \frac{z_1 \cdot z}{z}$.

(67) If $z_1 \neq 0_{\mathbb{C}_F}$ and $z_2 \neq 0_{\mathbb{C}_F}$, then $(\frac{z_1}{z_2})^{-1} = \frac{z_2}{z_1}$.

(68) If $z_1 \neq 0_{\mathbb{C}_F}$ and $z_2 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1^{-1}}{z_2^{-1}} = \frac{z_2}{z_1}$.

(69) If $z_2 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2^{-1}} = z_1 \cdot z_2$.

(70) If $z_1 \neq 0_{\mathbb{C}_F}$ and $z_2 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1^{-1}}{z_2} = (z_1 \cdot z_2)^{-1}$.

(71) If $z_1 \neq 0_{\mathbb{C}_F}$ and $z_2 \neq 0_{\mathbb{C}_F}$, then $z_1^{-1} \cdot \frac{z}{z_2} = \frac{z}{z_1 \cdot z_2}$.

(72) If $z \neq 0_{\mathbb{C}_F}$ and $z_2 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2} = \frac{z_1 \cdot z}{z_2 \cdot z}$ and $\frac{z_1}{z_2} = \frac{z \cdot z_1}{z \cdot z_2}$.

(73) If $z_2 \neq 0_{\mathbb{C}_F}$ and $z_3 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2 \cdot z_3} = \frac{\frac{z_1}{z_2}}{z_3}$.

(74) If $z_2 \neq 0_{\mathbb{C}_F}$ and $z_3 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1 \cdot z_3}{z_2} = \frac{z_1}{\frac{z_2}{z_3}}$.

(75) If $z_2 \neq 0_{\mathbb{C}_F}$ and $z_3 \neq 0_{\mathbb{C}_F}$ and $z_4 \neq 0_{\mathbb{C}_F}$, then $\frac{\frac{z_1}{z_2}}{\frac{z_3}{z_4}} = \frac{z_1 \cdot z_4}{z_2 \cdot z_3}$.

(76) If $z_2 \neq 0_{\mathbb{C}_F}$ and $z_4 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 + z_3 \cdot z_2}{z_2 \cdot z_4}$.

(77) If $z \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z} + \frac{z_2}{z} = \frac{z_1 + z_2}{z}$.

(78) If $z_2 \neq 0_{\mathbb{C}_F}$, then $-\frac{z_1}{z_2} = \frac{-z_1}{z_2}$ and $-\frac{z_1}{z_2} = \frac{z_1}{-z_2}$.

(79) If $z_2 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2} = \frac{-z_1}{-z_2}$.

(80) If $z_2 \neq 0_{\mathbb{C}_F}$ and $z_4 \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z_2} - \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 - z_3 \cdot z_2}{z_2 \cdot z_4}$.

(81) If $z \neq 0_{\mathbb{C}_F}$, then $\frac{z_1}{z} - \frac{z_2}{z} = \frac{z_1 - z_2}{z}$.

(82) If $z_2 \neq 0_{\mathbb{C}_F}$ and if $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = \frac{z_3}{z_2}$.

(83) $\overline{\text{the zero of } \mathbb{C}_F} =$ the zero of $\mathbb{C}_F$.

(84) If $\overline{z} =$ the zero of $\mathbb{C}_F$, then $z =$ the zero of $\mathbb{C}_F$.

(85) $\overline{\text{the unity of } \mathbb{C}_F} =$ the unity of $\mathbb{C}_F$.

(86) $\overline{\overline{z}} = z$.

(87) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(88) $\overline{-z} = -\overline{z}$.

(89) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(90) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

(91) If $z \neq$ the zero of $\mathbb{C}_F$, then $\overline{z^{-1}} = \overline{z}^{-1}$.

(92) If $z_2 \neq$ the zero of $\mathbb{C}_F$, then $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$.

(93) $|\text{the zero of } \mathbb{C}_{\rm F}| = 0$.

(94) If $|z| = 0$, then $z = 0_{\mathbb{C}_{\rm F}}$.

(95) $0 \leq |z|$.

(96) $z \neq 0_{\mathbb{C}_{\rm F}}$ iff $0 < |z|$.

(97) $|\text{the unity of } \mathbb{C}_{\rm F}| = 1$.

(98) $|-z| = |z|$.

(99) $|\overline{z}| = |z|$.

(100) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(101) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(102) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(103) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(104) $|z_1 - z_2| = |z_2 - z_1|$.

(105) $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(106) $z_1 \neq z_2$ iff $0 < |z_1 - z_2|$.

(107) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

(108) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(109) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(110) If $z \neq$ the zero of $\mathbb{C}_{\rm F}$, then $|z^{-1}| = |z|^{-1}$.

(111) If $z_2 \neq$ the zero of $\mathbb{C}_{\rm F}$, then $\frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|$.

(112) $|z \cdot z| = |z \cdot \overline{z}|$.

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