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Inner Products and Angles of Complex Numbers

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Summary. An inner product of complex numbers is defined and used to characterize the (counter-clockwise) angle between (a,0) and (0,b) in the complex plane. For complex a, b and c we then define the (counter-clockwise) angle between (a,c) and (c, b) and prove theorems about the sum of internal and external angles of a triangle.

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The articles [10], [15], [12], [14], [16], [5], [9], [17], [7], [8], [1], [11], [3], [13], [4], [2], and [6] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following propositions are true:

(1) For all real numbers a, b holds $-(a+bi) = -a+(-b)i$.

(2) For all real numbers a, b such that $b > 0$ there exists a real number r such that $r = b \cdot -\lfloor \frac{a}{b} \rfloor + a$ and $0 \leq r$ and $r < b$.

(3) Let a, b, c be real numbers. Suppose $a > 0$ and $b \geq 0$ and $c \geq 0$ and $b < a$ and $c < a$. Let i be an integer. If $b = c + a \cdot i$, then $b = c$.

(4) For all real numbers a, b holds $\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$ and $\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$.

(5) For every real number a holds $\sin(a-\pi) = -\sin(a)$ and $\cos(a-\pi) = -\cos(a)$.

(6) For every real number a holds $\sin(a-\pi) = -\sin a$ and $\cos(a-\pi) = -\cos a$.

(7) For all real numbers a, b such that $a \in]0, \frac{\pi}{2}[$ and $b \in]0, \frac{\pi}{2}[$ holds $a < b$ iff $\sin a < \sin b$.

(8) For all real numbers a, b such that $a \in]\frac{\pi}{2}, \pi[$ and $b \in]\frac{\pi}{2}, \pi[$ holds $a < b$ iff $\sin a > \sin b$.

(9) For every real number a and for every integer i holds $\sin a = \sin(2 \cdot \pi \cdot i + a)$.

(10) For every real number a and for every integer i holds $\cos a = \cos(2 \cdot \pi \cdot i + a)$.

(11) For every real number a such that $\sin a = 0$ holds $\cos a \neq 0$.

(12) For all real numbers a, b such that $0 \leq a$ and $a < 2 \cdot \pi$ and $0 \leq b$ and $b < 2 \cdot \pi$ and $\sin a = \sin b$ and $\cos a = \cos b$ holds $a = b$.

2. More on the Argument of a Complex Number

Let us note that $\mathbb{C}_{\rm F}$ is non empty.

Let z be an element of $\mathbb{C}$. The functor Ftize(z) yielding an element of $\mathbb{C}_{\rm F}$ is defined by:

(Def. 1) $\mathrm{Ftize}(z) = z$.

We now state four propositions:

(13) For every element z of $\mathbb{C}$ holds $\Re(z) = \Re(\mathrm{Ftize}(z))$ and $\Im(z) = \Im(\mathrm{Ftize}(z))$.

(14) For all elements x, y of $\mathbb{C}$ holds $\mathrm{Ftize}(x+y) = \mathrm{Ftize}(x) + \mathrm{Ftize}(y)$.

(15) For every element z of $\mathbb{C}$ holds $z = 0_{\mathbb{C}}$ iff $\mathrm{Ftize}(z) = 0_{\mathbb{C}_{\rm F}}$.

(16) For every element z of $\mathbb{C}$ holds $|z| = |\mathrm{Ftize}(z)|$.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Arg}\, z$ yields a real number and is defined by:

(Def. 2) $\mathrm{Arg}\, z = \mathrm{Arg}\,\mathrm{Ftize}(z)$.

Next we state a number of propositions:

(17) For every element z of $\mathbb{C}$ and for every element u of $\mathbb{C}_{\rm F}$ such that $z = u$ holds $\mathrm{Arg}\, z = \mathrm{Arg}\, u$.

(18) For every element z of $\mathbb{C}$ holds $0 \leq \mathrm{Arg}\, z$ and $\mathrm{Arg}\, z < 2 \cdot \pi$.

(19) For every element z of $\mathbb{C}$ holds $z = |z| \cdot \cos \mathrm{Arg}\, z + (|z| \cdot \sin \mathrm{Arg}\, z)i$.

(20) $\mathrm{Arg}(0_{\mathbb{C}}) = 0$.

(21) Let z be an element of $\mathbb{C}$ and r be a real number. If $z \neq 0$ and $z = |z| \cdot \cos r + (|z| \cdot \sin r)i$ and $0 \leq r$ and $r < 2 \cdot \pi$, then $r = \mathrm{Arg}\, z$.

(22) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds if $\mathrm{Arg}\, z < \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z + \pi$ and if $\mathrm{Arg}\, z \geq \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z - \pi$.

(23) For every real number r such that $r \geq 0$ holds $\mathrm{Arg}(r + 0i) = 0$.

(24) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $z = |z| + 0i$.

(25) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds $\mathrm{Arg}\, z < \pi$ iff $\mathrm{Arg}(-z) \geq \pi$.

(26) For all elements x, y of $\mathbb{C}$ such that $x \neq y$ or $x - y \neq 0_{\mathbb{C}}$ holds $\mathrm{Arg}(x-y) < \pi$ iff $\mathrm{Arg}(y-x) \geq \pi$.

(27) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]0, \pi[$ iff $\Im(z) > 0$.

(28) For every element z of $\mathbb{C}$ such that $\mathrm{Arg}\, z \neq 0$ holds $\mathrm{Arg}\, z < \pi$ iff $\sin \mathrm{Arg}\, z > 0$.

(29) For all elements x, y of $\mathbb{C}$ such that $\mathrm{Arg}\, x < \pi$ and $\mathrm{Arg}\, y < \pi$ holds $\mathrm{Arg}(x+y) < \pi$.

(30) For every real number x such that $x > 0$ holds $\mathrm{Arg}(0 + xi) = \frac{\pi}{2}$.

(31) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(32) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(33) For every element z of $\mathbb{C}$ such that $\Im(z) > 0$ holds $\sin \mathrm{Arg}\, z > 0$.

(34) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $\Re(z) \geq 0$ and $\Im(z) = 0$.

(35) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = \pi$ iff $\Re(z) < 0$ and $\Im(z) = 0$.

(36) For every element z of $\mathbb{C}$ holds $\Im(z) = 0$ iff $\mathrm{Arg}\, z = 0$ or $\mathrm{Arg}\, z = \pi$.

(37) For every element z of $\mathbb{C}$ such that $\operatorname{Arg} z \leq \pi$ holds $\Im(z) \geq 0$.

(38) For every element z of $\mathbb{C}$ such that $z \neq 0$ holds $\cos \operatorname{Arg}(-z) = -\cos \operatorname{Arg} z$ and $\sin \operatorname{Arg}(-z) = -\sin \operatorname{Arg} z$.

(39) For every element a of $\mathbb{C}$ such that $a \neq 0$ holds $\cos \operatorname{Arg} a = \frac{\Re(a)}{|a|}$ and $\sin \operatorname{Arg} a = \frac{\Im(a)}{|a|}$.

(40) For every element a of $\mathbb{C}$ and for every real number r such that $r > 0$ holds $\operatorname{Arg}(a \cdot (r + 0i)) = \operatorname{Arg} a$.

(41) For every element a of $\mathbb{C}$ and for every real number r such that $r < 0$ holds $\operatorname{Arg}(a \cdot (r + 0i)) = \operatorname{Arg}(-a)$.

3. Inner Product

Let x, y be elements of $\mathbb{C}$. The functor $(x|y)$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 3) $(x|y) = x \cdot \overline{y}$.

In the sequel a, b, c, d, x, y, z denote elements of $\mathbb{C}$.

The following propositions are true:

(42) $(x|y) = (\Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)) + (-\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y))i$.

(43) $(z|z) = (\Re(z) \cdot \Re(z) + \Im(z) \cdot \Im(z)) + 0i$ and $(z|z) = (\Re(z)^2 + \Im(z)^2) + 0i$.

(44) $(z|z) = |z|^2 + 0i$ and $|z|^2 = \Re((z|z))$.

(45) $|(x|y)| = |x| \cdot |y|$.

(46) If $(x|x) = 0$, then $x = 0$.

(47) $(y|x) = \overline{(x|y)}$.

(48) $((x+y)|z) = (x|z) + (y|z)$.

(49) $(x|(y+z)) = (x|y) + (x|z)$.

(50) $((a \cdot x)|y) = a \cdot (x|y)$.

(51) $(x|(a \cdot y)) = \overline{a} \cdot (x|y)$.

(52) $((a \cdot x)|y) = (x|(\overline{a} \cdot y))$.

(53) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.

(54) $(x|(a \cdot y + b \cdot z)) = \overline{a} \cdot (x|y) + \overline{b} \cdot (x|z)$.

(55) $((-x)|y) = (x|-y)$.

(56) $((-x)|y) = -(x|y)$.

(57) $-(x|y) = (x|-y)$.

(58) $((-x)|-y) = (x|y)$.

(59) $((x-y)|z) = (x|z) - (y|z)$.

(60) $(x|(y-z)) = (x|y) - (x|z)$.

(61) $(0_{\mathbb{C}}|x) = 0_{\mathbb{C}}$ and $(x|0_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(62) $((x+y)|(x+y)) = (x|x) + (x|y) + (y|x) + (y|y)$.

(63) $((x-y)|(x-y)) = ((x|x) - (x|y) - (y|x)) + (y|y)$.

(64) $\Re((x|y)) = 0$ iff $\Im((x|y)) = |x| \cdot |y|$ or $\Im((x|y)) = -|x| \cdot |y|$.

4. Rotation

Let a be an element of $\mathbb{C}$ and let r be a real number. The functor $a \circlearrowleft r$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 4) $a \circlearrowleft r = |a| \cdot \cos(r + \operatorname{Arg} a) + (|a| \cdot \sin(r + \operatorname{Arg} a))i$.

In the sequel r is a real number.

We now state a number of propositions:

(65) $a \circlearrowleft 0 = a$.

(66) $a \circlearrowleft r = 0_{\mathbb{C}}$ iff $a = 0_{\mathbb{C}}$.

(67) $|a \circlearrowleft r| = |a|$.

(68) If $a \neq 0_{\mathbb{C}}$, then there exists an integer i such that $\operatorname{Arg}(a \circlearrowleft r) = 2 \cdot \pi \cdot i + (r + \operatorname{Arg} a)$.

(69) $a \circlearrowleft -\operatorname{Arg} a = |a| + 0i$.

(70) $\Re(a \circlearrowleft r) = \Re(a) \cdot \cos r - \Im(a) \cdot \sin r$ and $\Im(a \circlearrowleft r) = \Re(a) \cdot \sin r + \Im(a) \cdot \cos r$.

(71) $a + b \circlearrowleft r = (a \circlearrowleft r) + (b \circlearrowleft r)$.

(72) $-a \circlearrowleft r = -(a \circlearrowleft r)$.

(73) $a - b \circlearrowleft r = (a \circlearrowleft r) - (b \circlearrowleft r)$.

(74) $a \circlearrowleft \pi = -a$.

5. Angles

Let a, b be elements of $\mathbb{C}$. The functor $\measuredangle(a,b)$ yielding a real number is defined as follows:

(Def. 5) $\measuredangle(a,b) = \begin{cases} \operatorname{Arg}(b \circlearrowleft -\operatorname{Arg} a), & \text{if } \operatorname{Arg} a = 0 \text{ or } b \neq 0, \\ 2 \cdot \pi - \operatorname{Arg} a, & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(75) If $r \geq 0$, then $\measuredangle(r + 0i, a) = \operatorname{Arg} a$.

(76) If $\operatorname{Arg} a = \operatorname{Arg} b$ and $a \neq 0$ and $b \neq 0$, then $\operatorname{Arg}(a \circlearrowleft r) = \operatorname{Arg}(b \circlearrowleft r)$.

(77) If $r > 0$, then $\measuredangle(a,b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(78) If $a \neq 0$ and $b \neq 0$ and $\operatorname{Arg} a = \operatorname{Arg} b$, then $\operatorname{Arg}(-a) = \operatorname{Arg}(-b)$.

(79) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r)$.

(80) If $r < 0$ and $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(81) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a,b) = \measuredangle(-a,-b)$.

(82) If $b \neq 0$ and $\measuredangle(a,b) = 0$, then $\measuredangle(a,-b) = \pi$.

(83) If $a \neq 0$ and $b \neq 0$, then $\cos \measuredangle(a,b) = \frac{\Re((a|b))}{|a| \cdot |b|}$ and $\sin \measuredangle(a,b) = -\frac{\Im((a|b))}{|a| \cdot |b|}$.

Let x, y, z be elements of $\mathbb{C}$. The functor $\measuredangle(x,y,z)$ yields a real number and is defined as follows:

(Def. 6) $\measuredangle(x,y,z) = \begin{cases} \operatorname{Arg}(z - y) - \operatorname{Arg}(x - y), & \text{if } \operatorname{Arg}(z - y) - \operatorname{Arg}(x - y) \geq 0, \\ 2 \cdot \pi + (\operatorname{Arg}(z - y) - \operatorname{Arg}(x - y)), & \text{otherwise.} \end{cases}$

Next we state a number of propositions:

(84) $0 \leq \measuredangle(x,y,z)$ and $\measuredangle(x,y,z) < 2 \cdot \pi$.

(85) $\measuredangle(x,y,z) = \measuredangle(x-y,0_{\mathbb{C}},z-y)$.

(86) $\measuredangle(a,b,c) = \measuredangle(a+d,b+d,c+d)$.

(87) $\measuredangle(a,b) = \measuredangle(a,0_{\mathbb{C}},b)$.

(88) If $\measuredangle(x,y,z) = 0$, then $\mathrm{Arg}(x-y) = \mathrm{Arg}(z-y)$ and $\measuredangle(z,y,x) = 0$.

(89) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Re((a|b)) = 0$ iff $\measuredangle(a,0_{\mathbb{C}},b) = \frac{\pi}{2}$ or $\measuredangle(a,0_{\mathbb{C}},b) = \frac{3}{2}\cdot\pi$.

(90) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Im((a|b)) = |a|\cdot|b|$ or $\Im((a|b)) = -|a|\cdot|b|$ iff $\measuredangle(a,0_{\mathbb{C}},b) = \frac{\pi}{2}$ or $\measuredangle(a,0_{\mathbb{C}},b) = \frac{3}{2}\cdot\pi$.

(91) If $x \neq y$ and if $z \neq y$ and if $\measuredangle(x,y,z) = \frac{\pi}{2}$ or $\measuredangle(x,y,z) = \frac{3}{2}\cdot\pi$, then $|x-y|^{\mathbf{2}} + |z-y|^{\mathbf{2}} = |x-z|^{\mathbf{2}}$.

(92) If $a \neq b$ and $b \neq c$, then $\measuredangle(a,b,c) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r, c \circlearrowleft r)$.

(93) $\measuredangle(a,b,a) = 0$.

(94) $\measuredangle(a,b,c) \neq 0$ iff $\measuredangle(a,b,c) + \measuredangle(c,b,a) = 2\cdot\pi$.

(95) If $\measuredangle(a,b,c) \neq 0$, then $\measuredangle(c,b,a) \neq 0$.

(96) If $\measuredangle(a,b,c) = \pi$, then $\measuredangle(c,b,a) = \pi$.

(97) If $a \neq b$ and $a \neq c$ and $b \neq c$, then $\measuredangle(a,b,c) \neq 0$ or $\measuredangle(b,c,a) \neq 0$ or $\measuredangle(c,a,b) \neq 0$.

(98) If $a \neq b$ and $b \neq c$ and $0 < \measuredangle(a,b,c)$ and $\measuredangle(a,b,c) < \pi$, then $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = \pi$ and $0 < \measuredangle(b,c,a)$ and $0 < \measuredangle(c,a,b)$.

(99) If $a \neq b$ and $b \neq c$ and $\measuredangle(a,b,c) > \pi$, then $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = 5\cdot\pi$ and $\measuredangle(b,c,a) > \pi$ and $\measuredangle(c,a,b) > \pi$.

(100) If $a \neq b$ and $b \neq c$ and $\measuredangle(a,b,c) = \pi$, then $\measuredangle(b,c,a) = 0$ and $\measuredangle(c,a,b) = 0$.

(101) If $a \neq b$ and $a \neq c$ and $b \neq c$ and $\measuredangle(a,b,c) = 0$, then $\measuredangle(b,c,a) = 0$ and $\measuredangle(c,a,b) = \pi$ or $\measuredangle(b,c,a) = \pi$ and $\measuredangle(c,a,b) = 0$.

(102) $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = \pi$ or $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = 5\cdot\pi$ iff $a \neq b$ and $a \neq c$ and $b \neq c$.

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