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The Collinearity Structure

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Summary. The text includes basic axioms and theorems concerning the collinearity structure based on Wanda Szmielew [2], pp. 18–20. Collinearity is defined as a relation on Cartesian product $[:S,S,S:]$ of set S. The basic text is preceeded with a few auxiliary theorems (e.g: ternary relation). Then come the two basic axioms of the collinearity structure: A1.1.1 and A1.1.2 and a few theorems. Another axiom: Aks dim, which states that there exist at least 3 non-collinear points, excludes the trivial structures (i.e. pairs $\langle S, [:S,S,S:]\rangle$). Following it the notion of a line is included and several additional theorems are appended.

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The articles [4], [1], [5], and [3] provide the notation and terminology for this paper.

In this paper X is a set.

Let us consider X. A set is called a 3-ary relation of X if:

(Def. 1) It $\subseteq [:X,X,X:]$.

The following proposition is true

(2)[1] $X=\emptyset$ or there exists a set a such that $\{a\}=X$ or there exist sets a, b such that $a\neq b$ and $a\in X$ and $b\in X$.

We consider collinearity structures as extensions of 1-sorted structure as systems
⟨ a carrier, a collinearity relation ⟩,
where the carrier is a set and the collinearity relation is a 3-ary relation of the carrier.

Let us observe that there exists a collinearity structure which is non empty and strict.

In the sequel C_1 is a non empty collinearity structure and a, b, c are points of C_1.

Let us consider C_1, a, b, c. We say that a, b and c are collinear if and only if:

(Def. 2) $\langle a,b,c\rangle\in$ the collinearity relation of C_1.

Let I_1 be a non empty collinearity structure. We say that I_1 is reflexive if and only if:

(Def. 3) For all points a, b, c of I_1 such that $a=b$ or $a=c$ or $b=c$ holds $\langle a,b,c\rangle\in$ the collinearity relation of I_1.

Let I_1 be a non empty collinearity structure. We say that I_1 is transitive if and only if the condition (Def. 4) is satisfied.

[1] The proposition (1) has been removed.

(Def. 4) Let a, b, p, q, r be points of I_1. Suppose that

(i) $a \neq b$,

(ii) $\langle a, b, p \rangle \in$ the collinearity relation of I_1,

(iii) $\langle a, b, q \rangle \in$ the collinearity relation of I_1, and

(iv) $\langle a, b, r \rangle \in$ the collinearity relation of I_1.

Then $\langle p, q, r \rangle \in$ the collinearity relation of I_1.

Let us note that there exists a non empty collinearity structure which is strict, reflexive, and transitive.

A collinearity space is a reflexive transitive non empty collinearity structure.

We follow the rules: C_2 is a collinearity space and a, b, c, d, p, q, r are points of C_2.

Next we state several propositions:

(7)[2] If $a = b$ or $a = c$ or $b = c$, then a, b and c are collinear.

(8) Suppose $a \neq b$ and a, b and p are collinear and a, b and q are collinear and a, b and r are collinear. Then p, q and r are collinear.

(9) If a, b and c are collinear, then b, a and c are collinear and a, c and b are collinear.

(10) a, b and a are collinear.

(11) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

(12) If a, b and c are collinear, then b, a and c are collinear.

(13) If a, b and c are collinear, then b, c and a are collinear.

(14) Suppose $p \neq q$ and a, b and p are collinear and a, b and q are collinear and p, q and r are collinear. Then a, b and r are collinear.

Let us consider C_2, a, b. The functor $\mathrm{Line}(a,b)$ yields a set and is defined as follows:

(Def. 5) $\mathrm{Line}(a,b) = \{p : a,\ b \text{ and } p \text{ are collinear}\}$.

We now state two propositions:

(16)[3] $a \in \mathrm{Line}(a,b)$ and $b \in \mathrm{Line}(a,b)$.

(17) a, b and r are collinear iff $r \in \mathrm{Line}(a,b)$.

Let I_1 be a non empty collinearity structure. We say that I_1 is proper if and only if:

(Def. 6) There exist points a, b, c of I_1 such that a, b and c are not collinear.

One can check that there exists a collinearity space which is strict and proper.

We use the following convention: C_2 denotes a proper collinearity space and a, b, p, q, r denote points of C_2.

One can prove the following proposition

(19)[4] For all p, q such that $p \neq q$ there exists r such that p, q and r are not collinear.

Let us consider C_2. A set is called a line of C_2 if:

(Def. 7) There exist a, b such that $a \neq b$ and it $= \mathrm{Line}(a,b)$.

In the sequel P, Q denote lines of C_2.

We now state a number of propositions:

[2] The propositions (3)–(6) have been removed.
[3] The proposition (15) has been removed.
[4] The proposition (18) has been removed.

(22)[5] If $a = b$, then $\mathrm{Line}(a, b) =$ the carrier of C_2.

(23) For every P there exist a, b such that $a \neq b$ and $a \in P$ and $b \in P$.

(24) If $a \neq b$, then there exists P such that $a \in P$ and $b \in P$.

(25) If $p \in P$ and $q \in P$ and $r \in P$, then p, q and r are collinear.

(26) If $P \subseteq Q$, then $P = Q$.

(27) If $p \neq q$ and $p \in P$ and $q \in P$, then $\mathrm{Line}(p, q) \subseteq P$.

(28) If $p \neq q$ and $p \in P$ and $q \in P$, then $\mathrm{Line}(p, q) = P$.

(29) If $p \neq q$ and $p \in P$ and $q \in P$ and $p \in Q$ and $q \in Q$, then $P = Q$.

(30) $P = Q$ or P misses Q or there exists p such that $P \cap Q = \{p\}$.

(31) If $a \neq b$, then $\mathrm{Line}(a, b) \neq$ the carrier of C_2.

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[5] The propositions (20) and (21) have been removed.