

Continuous, Stable, and Linear Maps of Coherence Spaces

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MML Identifier: COHSP_1.

WWW: http://mizar.org/JFM/Vol7/cohsp_1.html

The articles [14], [13], [9], [17], [15], [18], [6], [5], [10], [16], [4], [11], [1], [2], [12], [3], [8], and [7] provide the notation and terminology for this paper.

1. DIRECTED SETS

One can verify that there exists a coherent space which is finite. Let X be a set. Let us observe that X is binary complete if and only if:

(Def. 1) For every set A such that for all sets a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in X$ holds $\bigcup A \in X$.

Let X be a set. The functor $\text{FlatCoh}(X)$ yields a set and is defined as follows:

(Def. 2) $\text{FlatCoh}(X) = \text{CohSp}(\text{id}_X)$.

The functor $\text{SubFin}(X)$ yielding a subset of X is defined by:

(Def. 3) For every set x holds $x \in \text{SubFin}(X)$ iff $x \in X$ and x is finite.

Next we state three propositions:

- (1) For all sets X , x holds $x \in \text{FlatCoh}(X)$ iff $x = \emptyset$ or there exists a set y such that $x = \{y\}$ and $y \in X$.
- (2) For every set X holds $\bigcup \text{FlatCoh}(X) = X$.
- (3) For every finite subset-closed set X holds $\text{SubFin}(X) = X$.

One can verify that $\{\emptyset\}$ is subset-closed and binary complete. Let X be a set. Observe that 2^X is subset-closed and binary complete and $\text{FlatCoh}(X)$ is non empty, subset-closed, and binary complete.

Let C be a non empty subset-closed set. One can check that $\text{SubFin}(C)$ is non empty and subset-closed.

One can prove the following proposition

- (4) $\text{Web}(\{\emptyset\}) = \emptyset$.

The scheme *MinimalElement wrt Incl* deals with sets \mathcal{A} , \mathcal{B} and a unary predicate \mathcal{P} , and states that:

There exists a set a such that $a \in \mathcal{B}$ and $\mathcal{P}[a]$ and for every set b such that $b \in \mathcal{B}$ and $\mathcal{P}[b]$ and $b \subseteq a$ holds $b = a$

provided the following requirements are met:

- $\mathcal{A} \in \mathcal{B}$,
- $\mathcal{P}[\mathcal{A}]$, and
- \mathcal{A} is finite.

Let C be a coherent space. One can check that there exists an element of C which is finite.

Let X be a set. We say that X is \cup -directed if and only if:

(Def. 4) For every finite subset Y of X there exists a set a such that $\bigcup Y \subseteq a$ and $a \in X$.

We say that X is \cap -directed if and only if:

(Def. 5) For every finite subset Y of X there exists a set a such that for every set y such that $y \in Y$ holds $a \subseteq y$ and $a \in X$.

Let us mention that every set which is \cup -directed is also non empty and every set which is \cap -directed is also non empty.

One can prove the following propositions:

- (5) Let X be a set. Suppose X is \cup -directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $a \cup b \subseteq c$ and $c \in X$.
- (6) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $a \cup b \subseteq c$ and $c \in X$. Then X is \cup -directed.
- (7) Let X be a set. Suppose X is \cap -directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $c \subseteq a \cap b$ and $c \in X$.
- (8) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $c \subseteq a \cap b$ and $c \in X$. Then X is \cap -directed.
- (9) For every set x holds $\{x\}$ is \cup -directed and \cap -directed.
- (10) For all sets x, y holds $\{x, y, x \cup y\}$ is \cup -directed.
- (11) For all sets x, y holds $\{x, y, x \cap y\}$ is \cap -directed.

Let us note that there exists a set which is \cup -directed, \cap -directed, and finite.

Let C be a non empty set. Note that there exists a subset of C which is \cup -directed, \cap -directed, and finite.

We now state the proposition

(12) For every set X holds $\text{Fin}X$ is \cup -directed and \cap -directed.

Let X be a set. Observe that $\text{Fin}X$ is \cup -directed and \cap -directed.

Let C be a subset-closed non empty set. Observe that there exists a subset of C which is preboolean and non empty.

Let C be a subset-closed non empty set and let a be an element of C . Then $\text{Fin}a$ is a preboolean non empty subset of C .

We now state the proposition

(13) Let X be a non empty set and Y be a set. Suppose X is \cup -directed and $Y \subseteq \bigcup X$ and Y is finite. Then there exists a set Z such that $Z \in X$ and $Y \subseteq Z$.

Let X be a set. We introduce X is multiplicative as a synonym of X is \cap -closed.

Let X be a set. We say that X is closed under directed unions if and only if:

(Def. 7)¹ For every subset A of X such that A is \cup -directed holds $\cup A \in X$.

Let us observe that every set which is subset-closed is also multiplicative.
One can prove the following proposition

(15)² For every coherent space C and for every \cup -directed subset A of C holds $\cup A \in C$.

Let us mention that every coherent space is closed under directed unions.
One can check that there exists a coherent space which is multiplicative and closed under directed unions.

Let C be a closed under directed unions non empty set and let A be a \cup -directed subset of C .
Then $\cup A$ is an element of C .

Let X, Y be sets. We say that X includes lattice of Y if and only if:

(Def. 8) For all sets a, b such that $a \in Y$ and $b \in Y$ holds $a \cap b \in X$ and $a \cup b \in X$.

Next we state the proposition

(16) For every non empty set X such that X includes lattice of X holds X is \cup -directed and \cap -directed.

Let X, x, y be sets. We say that X includes lattice of x, y if and only if:

(Def. 9) X includes lattice of $\{x, y\}$.

Next we state the proposition

(17) For all sets X, x, y holds X includes lattice of x, y iff $x \in X$ and $y \in X$ and $x \cap y \in X$ and $x \cup y \in X$.

2. CONTINUOUS, STABLE, AND LINEAR FUNCTIONS

Let f be a function. We say that f is preserving arbitrary unions if and only if:

(Def. 10) For every subset A of $\text{dom } f$ such that $\cup A \in \text{dom } f$ holds $f(\cup A) = \cup(f^\circ A)$.

We say that f is preserving directed unions if and only if:

(Def. 11) For every subset A of $\text{dom } f$ such that A is \cup -directed and $\cup A \in \text{dom } f$ holds $f(\cup A) = \cup(f^\circ A)$.

Let f be a function. We say that f is \subseteq -monotone if and only if:

(Def. 12) For all sets a, b such that $a \in \text{dom } f$ and $b \in \text{dom } f$ and $a \subseteq b$ holds $f(a) \subseteq f(b)$.

We say that f is preserving binary intersections if and only if:

(Def. 13) For all sets a, b such that $\text{dom } f$ includes lattice of a, b holds $f(a \cap b) = f(a) \cap f(b)$.

Let us observe that every function which is preserving directed unions is also \subseteq -monotone and every function which is preserving arbitrary unions is also preserving directed unions.

We now state two propositions:

(18) Let f be a function. Suppose f is preserving arbitrary unions. Let x, y be sets. If $x \in \text{dom } f$ and $y \in \text{dom } f$ and $x \cup y \in \text{dom } f$, then $f(x \cup y) = f(x) \cup f(y)$.

(19) For every function f such that f is preserving arbitrary unions holds $f(\emptyset) = \emptyset$.

¹ The definition (Def. 6) has been removed.

² The proposition (14) has been removed.

Let C_1, C_2 be coherent spaces. Note that there exists a function from C_1 into C_2 which is preserving arbitrary unions and preserving binary intersections.

Let C be a coherent space. Note that there exists a many sorted set indexed by C which is preserving arbitrary unions and preserving binary intersections.

Let f be a function. We say that f is continuous if and only if:

(Def. 14) $\text{dom } f$ is closed under directed unions and f is preserving directed unions.

Let f be a function. We say that f is stable if and only if:

(Def. 15) $\text{dom } f$ is multiplicative and f is continuous and preserving binary intersections.

Let f be a function. We say that f is linear if and only if:

(Def. 16) f is stable and preserving arbitrary unions.

One can verify the following observations:

- * every function which is continuous is also preserving directed unions,
- * every function which is stable is also preserving binary intersections and continuous, and
- * every function which is linear is also preserving arbitrary unions and stable.

Let X be a closed under directed unions set. Note that every many sorted set indexed by X which is preserving directed unions is also continuous.

Let X be a multiplicative set. Observe that every many sorted set indexed by X which is continuous and preserving binary intersections is also stable.

One can check that every function which is stable and preserving arbitrary unions is also linear.

Note that there exists a function which is linear. Let C be a coherent space. Note that there exists a many sorted set indexed by C which is linear. Let B be a coherent space. One can check that there exists a function from B into C which is linear.

Let f be a continuous function. Observe that $\text{dom } f$ is closed under directed unions.

Let f be a stable function. Note that $\text{dom } f$ is multiplicative.

Next we state several propositions:

- (20) For every set X holds $\bigcup \text{Fin } X = X$.
- (21) For every continuous function f such that $\text{dom } f$ is subset-closed and for every set a such that $a \in \text{dom } f$ holds $f(a) = \bigcup (f^\circ \text{Fin } a)$.
- (22) Let f be a function. Suppose $\text{dom } f$ is subset-closed. Then f is continuous if and only if the following conditions are satisfied:
- (i) $\text{dom } f$ is closed under directed unions,
 - (ii) f is \subseteq -monotone, and
 - (iii) for all sets a, y such that $a \in \text{dom } f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$.
- (23) Let f be a function. Suppose $\text{dom } f$ is subset-closed and closed under directed unions. Then f is stable if and only if the following conditions are satisfied:
- (i) f is \subseteq -monotone, and
 - (ii) for all sets a, y such that $a \in \text{dom } f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$ and for every set c such that $c \subseteq a$ and $y \in f(c)$ holds $b \subseteq c$.
- (24) Let f be a function. Suppose $\text{dom } f$ is subset-closed and closed under directed unions. Then f is linear if and only if the following conditions are satisfied:
- (i) f is \subseteq -monotone, and
 - (ii) for all sets a, y such that $a \in \text{dom } f$ and $y \in f(a)$ there exists a set x such that $x \in a$ and $y \in f(\{x\})$ and for every set b such that $b \subseteq a$ and $y \in f(b)$ holds $x \in b$.

3. GRAPH OF CONTINUOUS FUNCTION

Let f be a function. The functor $\text{graph}(f)$ yielding a set is defined as follows:

(Def. 17) For every set x holds $x \in \text{graph}(f)$ iff there exists a finite set y and there exists a set z such that $x = \langle y, z \rangle$ and $y \in \text{dom } f$ and $z \in f(y)$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2 . Then $\text{graph}(f)$ is a subset of $[:C_1, \cup C_2:]$.

Let f be a function. One can check that $\text{graph}(f)$ is relation-like.

We now state several propositions:

- (25) For every function f and for all sets x, y holds $\langle x, y \rangle \in \text{graph}(f)$ iff x is finite and $x \in \text{dom } f$ and $y \in f(x)$.
- (26) Let f be a \subseteq -monotone function and a, b be sets. Suppose $b \in \text{dom } f$ and $a \subseteq b$ and b is finite. Let y be a set. If $\langle a, y \rangle \in \text{graph}(f)$, then $\langle b, y \rangle \in \text{graph}(f)$.
- (27) Let C_1, C_2 be coherent spaces, f be a function from C_1 into C_2 , a be an element of C_1 , and y_1, y_2 be sets. If $\langle a, y_1 \rangle \in \text{graph}(f)$ and $\langle a, y_2 \rangle \in \text{graph}(f)$, then $\{y_1, y_2\} \in C_2$.
- (28) Let C_1, C_2 be coherent spaces, f be a \subseteq -monotone function from C_1 into C_2 , and a, b be elements of C_1 . Suppose $a \cup b \in C_1$. Let y_1, y_2 be sets. If $\langle a, y_1 \rangle \in \text{graph}(f)$ and $\langle b, y_2 \rangle \in \text{graph}(f)$, then $\{y_1, y_2\} \in C_2$.
- (29) For all coherent spaces C_1, C_2 and for all continuous functions f, g from C_1 into C_2 such that $\text{graph}(f) = \text{graph}(g)$ holds $f = g$.
- (30) Let C_1, C_2 be coherent spaces and X be a subset of $[:C_1, \cup C_2:]$. Suppose that
- (i) for every set x such that $x \in X$ holds x_1 is finite,
 - (ii) for all finite elements a, b of C_1 such that $a \subseteq b$ and for every set y such that $\langle a, y \rangle \in X$ holds $\langle b, y \rangle \in X$, and
 - (iii) for every finite element a of C_1 and for all sets y_1, y_2 such that $\langle a, y_1 \rangle \in X$ and $\langle a, y_2 \rangle \in X$ holds $\{y_1, y_2\} \in C_2$.
- Then there exists a continuous function f from C_1 into C_2 such that $X = \text{graph}(f)$.
- (31) Let C_1, C_2 be coherent spaces, f be a continuous function from C_1 into C_2 , and a be an element of C_1 . Then $f(a) = (\text{graph}(f))^\circ \text{Fin } a$.

4. TRACE OF STABLE FUNCTION

Let f be a function. The functor $\text{Trace}(f)$ yielding a set is defined by the condition (Def. 18).

(Def. 18) Let x be a set. Then $x \in \text{Trace}(f)$ if and only if there exist sets a, y such that $x = \langle a, y \rangle$ and $a \in \text{dom } f$ and $y \in f(a)$ and for every set b such that $b \in \text{dom } f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

One can prove the following proposition

- (32) Let f be a function and a, y be sets. Then $\langle a, y \rangle \in \text{Trace}(f)$ if and only if the following conditions are satisfied:
- (i) $a \in \text{dom } f$,
 - (ii) $y \in f(a)$, and
 - (iii) for every set b such that $b \in \text{dom } f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2 . Then $\text{Trace}(f)$ is a subset of $[:C_1, \cup C_2:]$.

Let f be a function. One can check that $\text{Trace}(f)$ is relation-like.

Next we state a number of propositions:

- (33) For every continuous function f such that $\text{dom } f$ is subset-closed holds $\text{Trace}(f) \subseteq \text{graph}(f)$.
- (34) Let f be a continuous function. Suppose $\text{dom } f$ is subset-closed. Let a, y be sets. If $\langle a, y \rangle \in \text{Trace}(f)$, then a is finite.
- (35) Let C_1, C_2 be coherent spaces, f be a \subseteq -monotone function from C_1 into C_2 , and a_1, a_2 be sets. Suppose $a_1 \cup a_2 \in C_1$. Let y_1, y_2 be sets. If $\langle a_1, y_1 \rangle \in \text{Trace}(f)$ and $\langle a_2, y_2 \rangle \in \text{Trace}(f)$, then $\{y_1, y_2\} \in C_2$.
- (36) Let C_1, C_2 be coherent spaces, f be a preserving binary intersections function from C_1 into C_2 , and a_1, a_2 be sets. If $a_1 \cup a_2 \in C_1$, then for every set y such that $\langle a_1, y \rangle \in \text{Trace}(f)$ and $\langle a_2, y \rangle \in \text{Trace}(f)$ holds $a_1 = a_2$.
- (37) Let C_1, C_2 be coherent spaces and f, g be stable functions from C_1 into C_2 . If $\text{Trace}(f) \subseteq \text{Trace}(g)$, then for every element a of C_1 holds $f(a) \subseteq g(a)$.
- (38) For all coherent spaces C_1, C_2 and for all stable functions f, g from C_1 into C_2 such that $\text{Trace}(f) = \text{Trace}(g)$ holds $f = g$.
- (39) Let C_1, C_2 be coherent spaces and X be a subset of $[C_1, \cup C_2]$. Suppose that
- (i) for every set x such that $x \in X$ holds x_1 is finite,
 - (ii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1 \rangle \in X$ and $\langle b, y_2 \rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and
 - (iii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for every set y such that $\langle a, y \rangle \in X$ and $\langle b, y \rangle \in X$ holds $a = b$.
- Then there exists a stable function f from C_1 into C_2 such that $X = \text{Trace}(f)$.
- (40) Let C_1, C_2 be coherent spaces, f be a stable function from C_1 into C_2 , and a be an element of C_1 . Then $f(a) = (\text{Trace}(f))^\circ \text{Fin } a$.
- (41) Let C_1, C_2 be coherent spaces, f be a stable function from C_1 into C_2 , a be an element of C_1 , and y be a set. Then $y \in f(a)$ if and only if there exists an element b of C_1 such that $\langle b, y \rangle \in \text{Trace}(f)$ and $b \subseteq a$.
- (42) For all coherent spaces C_1, C_2 there exists a stable function f from C_1 into C_2 such that $\text{Trace}(f) = \emptyset$.
- (43) Let C_1, C_2 be coherent spaces, a be a finite element of C_1 , and y be a set. If $y \in \cup C_2$, then there exists a stable function f from C_1 into C_2 such that $\text{Trace}(f) = \{\langle a, y \rangle\}$.
- (44) Let C_1, C_2 be coherent spaces, a be an element of C_1 , y be a set, and f be a stable function from C_1 into C_2 . Suppose $\text{Trace}(f) = \{\langle a, y \rangle\}$. Let b be an element of C_1 . Then
- (i) if $a \subseteq b$, then $f(b) = \{y\}$, and
 - (ii) if $a \not\subseteq b$, then $f(b) = \emptyset$.
- (45) Let C_1, C_2 be coherent spaces, f be a stable function from C_1 into C_2 , and X be a subset of $\text{Trace}(f)$. Then there exists a stable function g from C_1 into C_2 such that $\text{Trace}(g) = X$.
- (46) Let C_1, C_2 be coherent spaces and A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a stable function f from C_1 into C_2 such that $x \cup y = \text{Trace}(f)$. Then there exists a stable function f from C_1 into C_2 such that $\cup A = \text{Trace}(f)$.

Let C_1, C_2 be coherent spaces. The functor $\text{StabCoh}(C_1, C_2)$ yields a set and is defined as follows:

- (Def. 19) For every set x holds $x \in \text{StabCoh}(C_1, C_2)$ iff there exists a stable function f from C_1 into C_2 such that $x = \text{Trace}(f)$.

Let C_1, C_2 be coherent spaces. One can check that $\text{StabCoh}(C_1, C_2)$ is non empty, subset-closed, and binary complete.

One can prove the following propositions:

- (47) For all coherent spaces C_1, C_2 and for every stable function f from C_1 into C_2 holds $\text{Trace}(f) \subseteq [\text{SubFin}(C_1), \cup C_2]$.
- (48) For all coherent spaces C_1, C_2 holds $\cup \text{StabCoh}(C_1, C_2) = [\text{SubFin}(C_1), \cup C_2]$.
- (49) Let C_1, C_2 be coherent spaces, a, b be finite elements of C_1 , and y_1, y_2 be sets. Then $\langle \langle a, y_1 \rangle, \langle b, y_2 \rangle \rangle \in \text{Web}(\text{StabCoh}(C_1, C_2))$ if and only if one of the following conditions is satisfied:
 - (i) $a \cup b \notin C_1$ and $y_1 \in \cup C_2$ and $y_2 \in \cup C_2$, or
 - (ii) $\langle y_1, y_2 \rangle \in \text{Web}(C_2)$ and if $y_1 = y_2$, then $a = b$.

5. TRACE OF LINEAR FUNCTION

Next we state the proposition

- (50) Let C_1, C_2 be coherent spaces and f be a stable function from C_1 into C_2 . Then f is linear if and only if for all sets a, y such that $\langle a, y \rangle \in \text{Trace}(f)$ there exists a set x such that $a = \{x\}$.

Let f be a function. The functor $\text{LinTrace}(f)$ yielding a set is defined as follows:

- (Def. 20) For every set x holds $x \in \text{LinTrace}(f)$ iff there exist sets y, z such that $x = \langle y, z \rangle$ and $\langle \{y\}, z \rangle \in \text{Trace}(f)$.

Next we state three propositions:

- (51) For every function f and for all sets x, y holds $\langle x, y \rangle \in \text{LinTrace}(f)$ iff $\langle \{x\}, y \rangle \in \text{Trace}(f)$.
- (52) For every function f such that $f(\emptyset) = \emptyset$ and for all sets x, y such that $\{x\} \in \text{dom } f$ and $y \in f(\{x\})$ holds $\langle x, y \rangle \in \text{LinTrace}(f)$.
- (53) For every function f and for all sets x, y such that $\langle x, y \rangle \in \text{LinTrace}(f)$ holds $\{x\} \in \text{dom } f$ and $y \in f(\{x\})$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2 . Then $\text{LinTrace}(f)$ is a subset of $[\cup C_1, \cup C_2]$.

Let f be a function. Note that $\text{LinTrace}(f)$ is relation-like.

Let C_1, C_2 be coherent spaces. The functor $\text{LinCoh}(C_1, C_2)$ yielding a set is defined as follows:

- (Def. 21) For every set x holds $x \in \text{LinCoh}(C_1, C_2)$ iff there exists a linear function f from C_1 into C_2 such that $x = \text{LinTrace}(f)$.

One can prove the following propositions:

- (54) Let C_1, C_2 be coherent spaces, f be a \subseteq -monotone function from C_1 into C_2 , and x_1, x_2 be sets. Suppose $\{x_1, x_2\} \in C_1$. Let y_1, y_2 be sets. If $\langle x_1, y_1 \rangle \in \text{LinTrace}(f)$ and $\langle x_2, y_2 \rangle \in \text{LinTrace}(f)$, then $\{y_1, y_2\} \in C_2$.
- (55) Let C_1, C_2 be coherent spaces, f be a preserving binary intersections function from C_1 into C_2 , and x_1, x_2 be sets. If $\{x_1, x_2\} \in C_1$, then for every set y such that $\langle x_1, y \rangle \in \text{LinTrace}(f)$ and $\langle x_2, y \rangle \in \text{LinTrace}(f)$ holds $x_1 = x_2$.
- (56) For all coherent spaces C_1, C_2 and for all linear functions f, g from C_1 into C_2 such that $\text{LinTrace}(f) = \text{LinTrace}(g)$ holds $f = g$.

- (57) Let C_1, C_2 be coherent spaces and X be a subset of $[\cup C_1, \cup C_2]$. Suppose that
- (i) for all sets a, b such that $\{a, b\} \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1 \rangle \in X$ and $\langle b, y_2 \rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and
 - (ii) for all sets a, b such that $\{a, b\} \in C_1$ and for every set y such that $\langle a, y \rangle \in X$ and $\langle b, y \rangle \in X$ holds $a = b$.
- Then there exists a linear function f from C_1 into C_2 such that $X = \text{LinTrace}(f)$.
- (58) Let C_1, C_2 be coherent spaces, f be a linear function from C_1 into C_2 , and a be an element of C_1 . Then $f(a) = (\text{LinTrace}(f))^\circ a$.
- (59) For all coherent spaces C_1, C_2 there exists a linear function f from C_1 into C_2 such that $\text{LinTrace}(f) = \emptyset$.
- (60) Let C_1, C_2 be coherent spaces, x be a set, and y be a set. Suppose $x \in \cup C_1$ and $y \in \cup C_2$. Then there exists a linear function f from C_1 into C_2 such that $\text{LinTrace}(f) = \{\langle x, y \rangle\}$.
- (61) Let C_1, C_2 be coherent spaces, x be a set, and y be a set. Suppose $x \in \cup C_1$ and $y \in \cup C_2$. Let f be a linear function from C_1 into C_2 . Suppose $\text{LinTrace}(f) = \{\langle x, y \rangle\}$. Let a be an element of C_1 . Then
- (i) if $x \in a$, then $f(a) = \{y\}$, and
 - (ii) if $x \notin a$, then $f(a) = \emptyset$.
- (62) Let C_1, C_2 be coherent spaces, f be a linear function from C_1 into C_2 , and X be a subset of $\text{LinTrace}(f)$. Then there exists a linear function g from C_1 into C_2 such that $\text{LinTrace}(g) = X$.
- (63) Let C_1, C_2 be coherent spaces and A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a linear function f from C_1 into C_2 such that $x \cup y = \text{LinTrace}(f)$. Then there exists a linear function f from C_1 into C_2 such that $\cup A = \text{LinTrace}(f)$.

Let C_1, C_2 be coherent spaces. Note that $\text{LinCoh}(C_1, C_2)$ is non empty, subset-closed, and binary complete.

The following propositions are true:

- (64) For all coherent spaces C_1, C_2 holds $\cup \text{LinCoh}(C_1, C_2) = [\cup C_1, \cup C_2]$.
- (65) Let C_1, C_2 be coherent spaces, x_1, x_2 be sets, and y_1, y_2 be sets. Then $\langle \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \rangle \in \text{Web}(\text{LinCoh}(C_1, C_2))$ if and only if the following conditions are satisfied:
- (i) $x_1 \in \cup C_1$,
 - (ii) $x_2 \in \cup C_1$, and
 - (iii) $\langle x_1, x_2 \rangle \notin \text{Web}(C_1)$ and $y_1 \in \cup C_2$ and $y_2 \in \cup C_2$ or $\langle y_1, y_2 \rangle \in \text{Web}(C_2)$ and if $y_1 = y_2$, then $x_1 = x_2$.

6. NEGATION OF COHERENCE SPACES

Let C be a coherent space. The functor $\neg C$ yields a set and is defined by:

(Def. 22) $\neg C = \{a; a \text{ ranges over subsets of } \cup C : \bigwedge_{b: \text{element of } C} \bigvee_{x: \text{set}} a \cap b \subseteq \{x\}\}$.

The following proposition is true

- (66) Let C be a coherent space and x be a set. Then $x \in \neg C$ if and only if the following conditions are satisfied:
- (i) $x \subseteq \cup C$, and
 - (ii) for every element a of C there exists a set z such that $x \cap a \subseteq \{z\}$.

Let C be a coherent space. Observe that $\neg C$ is non empty, subset-closed, and binary complete. We now state several propositions:

- (67) For every coherent space C holds $\bigcup \neg C = \bigcup C$.
- (68) For every coherent space C and for all sets x, y such that $x \neq y$ and $\{x, y\} \in C$ holds $\{x, y\} \notin \neg C$.
- (69) For every coherent space C and for all sets x, y such that $\{x, y\} \subseteq \bigcup C$ and $\{x, y\} \notin C$ holds $\{x, y\} \in \neg C$.
- (70) For every coherent space C and for all sets x, y holds $\langle x, y \rangle \in \text{Web}(\neg C)$ iff $x \in \bigcup C$ but $y \in \bigcup C$ but $x = y$ or $\langle x, y \rangle \notin \text{Web}(C)$.
- (71) For every coherent space C holds $\neg \neg C = C$.
- (72) $\neg \{\emptyset\} = \{\emptyset\}$.
- (73) For every set X holds $\neg \text{FlatCoh}(X) = 2^X$ and $\neg(2^X) = \text{FlatCoh}(X)$.

7. PRODUCT AND COPRODUCT ON COHERENCE SPACES

Let x, y be sets. The functor $x \uplus y$ yields a set and is defined as follows:

(Def. 23) $x \uplus y = \bigcup \text{disjoint}\langle x, y \rangle$.

Next we state a number of propositions:

- (74) For all sets x, y holds $x \uplus y = [x, \{1\}] \cup [y, \{2\}]$.
- (75) For every set x holds $x \uplus \emptyset = [x, \{1\}]$ and $\emptyset \uplus x = [x, \{2\}]$.
- (76) For all sets x, y, z such that $z \in x \uplus y$ holds $z = \langle z_1, z_2 \rangle$ but $z_2 = 1$ and $z_1 \in x$ or $z_2 = 2$ and $z_1 \in y$.
- (77) For all sets x, y, z holds $\langle z, 1 \rangle \in x \uplus y$ iff $z \in x$.
- (78) For all sets x, y, z holds $\langle z, 2 \rangle \in x \uplus y$ iff $z \in y$.
- (79) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 \subseteq x_2 \uplus y_2$ iff $x_1 \subseteq x_2$ and $y_1 \subseteq y_2$.
- (80) For all sets x, y, z such that $z \subseteq x \uplus y$ there exist sets x_1, y_1 such that $z = x_1 \uplus y_1$ and $x_1 \subseteq x$ and $y_1 \subseteq y$.
- (81) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 = x_2 \uplus y_2$ iff $x_1 = x_2$ and $y_1 = y_2$.
- (82) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cup (x_2 \uplus y_2) = x_1 \cup x_2 \uplus y_1 \cup y_2$.
- (83) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cap (x_2 \uplus y_2) = x_1 \cap x_2 \uplus y_1 \cap y_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \sqcap C_2$ yielding a set is defined as follows:

(Def. 24) $C_1 \sqcap C_2 = \{a \uplus b : a \text{ ranges over elements of } C_1, b \text{ ranges over elements of } C_2\}$.

The functor $C_1 \sqcup C_2$ yields a set and is defined by:

(Def. 25) $C_1 \sqcup C_2 = \{a \uplus \emptyset : a \text{ ranges over elements of } C_1\} \cup \{\emptyset \uplus b : b \text{ ranges over elements of } C_2\}$.

Next we state several propositions:

- (84) Let C_1, C_2 be coherent spaces and x be a set. Then $x \in C_1 \sqcap C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$.
- (85) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcap C_2$ iff $x \in C_1$ and $y \in C_2$.
- (86) For all coherent spaces C_1, C_2 holds $\bigcup (C_1 \sqcap C_2) = \bigcup C_1 \uplus \bigcup C_2$.

- (87) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcup C_2$ iff $x \in C_1$ and $y = \emptyset$ or $x = \emptyset$ and $y \in C_2$.
- (88) Let C_1, C_2 be coherent spaces and x be a set. Suppose $x \in C_1 \sqcup C_2$. Then there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$ but $a = \emptyset$ or $b = \emptyset$.
- (89) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcup C_2) = \bigcup C_1 \uplus \bigcup C_2$.

Let C_1, C_2 be coherent spaces. Observe that $C_1 \sqcap C_2$ is non empty, subset-closed, and binary complete and $C_1 \sqcup C_2$ is non empty, subset-closed, and binary complete.

In the sequel C_1, C_2 are coherent spaces.

Next we state several propositions:

- (90) For all sets x, y holds $\langle \langle x, 1 \rangle, \langle y, 1 \rangle \rangle \in \text{Web}(C_1 \sqcap C_2)$ iff $\langle x, y \rangle \in \text{Web}(C_1)$.
- (91) For all sets x, y holds $\langle \langle x, 2 \rangle, \langle y, 2 \rangle \rangle \in \text{Web}(C_1 \sqcap C_2)$ iff $\langle x, y \rangle \in \text{Web}(C_2)$.
- (92) For all sets x, y such that $x \in \bigcup C_1$ and $y \in \bigcup C_2$ holds $\langle \langle x, 1 \rangle, \langle y, 2 \rangle \rangle \in \text{Web}(C_1 \sqcap C_2)$ and $\langle \langle y, 2 \rangle, \langle x, 1 \rangle \rangle \in \text{Web}(C_1 \sqcap C_2)$.
- (93) For all sets x, y holds $\langle \langle x, 1 \rangle, \langle y, 1 \rangle \rangle \in \text{Web}(C_1 \sqcup C_2)$ iff $\langle x, y \rangle \in \text{Web}(C_1)$.
- (94) For all sets x, y holds $\langle \langle x, 2 \rangle, \langle y, 2 \rangle \rangle \in \text{Web}(C_1 \sqcup C_2)$ iff $\langle x, y \rangle \in \text{Web}(C_2)$.
- (95) For all sets x, y holds $\langle \langle x, 1 \rangle, \langle y, 2 \rangle \rangle \notin \text{Web}(C_1 \sqcup C_2)$ and $\langle \langle y, 2 \rangle, \langle x, 1 \rangle \rangle \notin \text{Web}(C_1 \sqcup C_2)$.
- (96) $\neg(C_1 \sqcap C_2) = \neg C_1 \sqcup \neg C_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \otimes C_2$ yielding a set is defined by:

(Def. 26) $C_1 \otimes C_2 = \bigcup \{2^{[a, b]} : a \text{ ranges over elements of } C_1, b \text{ ranges over elements of } C_2\}$.

One can prove the following proposition

- (97) Let C_1, C_2 be coherent spaces and x be a set. Then $x \in C_1 \otimes C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x \subseteq [a, b]$.

Let C_1, C_2 be coherent spaces. Note that $C_1 \otimes C_2$ is non empty.

The following proposition is true

- (98) For all coherent spaces C_1, C_2 and for every element a of $C_1 \otimes C_2$ holds $\pi_1(a) \in C_1$ and $\pi_2(a) \in C_2$ and $a \subseteq [\pi_1(a), \pi_2(a)]$.

Let C_1, C_2 be coherent spaces. Note that $C_1 \otimes C_2$ is subset-closed and binary complete.

Next we state two propositions:

- (99) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \otimes C_2) = [\bigcup C_1, \bigcup C_2]$.
- (100) For all sets x_1, y_1, x_2, y_2 holds $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \text{Web}(C_1 \otimes C_2)$ iff $\langle x_1, y_1 \rangle \in \text{Web}(C_1)$ and $\langle x_2, y_2 \rangle \in \text{Web}(C_2)$.

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Received August 30, 1995

Published January 2, 2004
